Stochastic Processes, Ito Calculus and Black-Scholes formula

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1 Motivation

Already encountered Brownian motion

$$\dot{x}(t) = \xi(t) \tag{1}$$

with $\xi(t)$ a white Gaussian noise, i.e. $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$.

Also Langevin's equation

General question: how to treat noise in any differential equation of physics ? Examples:

- LRC circuit with external potential $V(t)$ which fluctuates in time:

$$L\dot{I}(t) + RI(t) + \frac{Q(t)}{C} = V(t) + \xi(t) \tag{2}$$

- harmonic oscillator with fluctuating frequency $\omega(t)$

$$\ddot{x} + \omega^2(t)x = 0 \tag{3}$$

- particle in a potential plus noise $\ddot{x} = -\frac{\partial H}{\partial x} + \xi$
- Ginzburg-Landau model with noise
- infinitely many more examples in physics, but also in economics, game theory etc etc.

This naturally leads to the concept of 'stochastic differential equations', i.e. ODEs and PDSs containing random coefficients or additive noise etc.

How to tackle such systems ? How to set up a 'calculus' for such problems ? Do rules of 'normal' (deterministic) calculus apply ? (the answer is no).

2 Examples

2.1 Random walk: diffusion equation

The random walk

$$\dot{x}(t) = \eta(t) \tag{4}$$

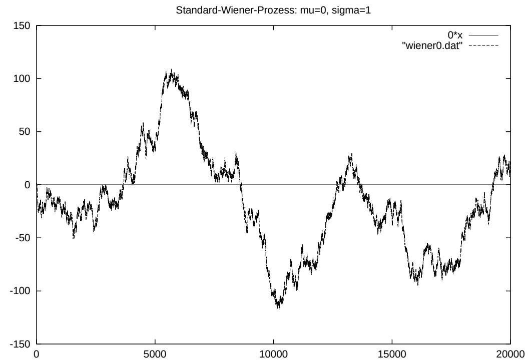


Figure 1: typical path of a Brownian motion ($2D = 1$)

with

$$\langle \eta(t)\eta(t') \rangle = 2D\delta(t - t') \tag{5}$$

leads to the diffusion equation: the density of particles at time t $P(x,t) = \langle \delta(x - x(t)) \rangle$ follows the partial differential equation

$$\partial_t P(x,t) = D\partial_x^2 P(x,t). \tag{6}$$

How do you check ? You know from (4) that $x(t)$ is a Gaussian random variable with mean zero at variance $2Dt$. This Gaussian distribution fulfills (6).

The process (4) is also referred to as the **Wiener process** (after N. Wiener).

2.2 Particles in a graviational field

Consider now a particle subject to a constant force and Gaussian white noise, i.e.

$$m\ddot{x} = -\gamma m\dot{x} - mg + \text{noise}. \tag{7}$$

Here g is the graviational force. In the overdamped case $\gamma\dot{x} \gg \ddot{x}$ we can write

$$\gamma m\dot{x} = -mg + \text{noise}, \tag{8}$$

i.e.

$$\dot{x}(t) = -\frac{g}{\gamma} + \xi(t) \tag{9}$$

with $\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t')$. Integration gives:

$$x(t) - x(0) = -\frac{g}{\gamma}t + \int_0^t \xi(t')dt' \tag{10}$$

i.e.

$$\langle x(t)\rangle = x(0) - \frac{g}{\gamma}t \tag{11}$$

and

$$\begin{aligned}\left\langle \left[x(t) - \left(x(0) - \frac{g}{\gamma}t\right)\right]^2 \right\rangle &= \int_0^t dt' \int_0^t dt'' \left\langle \xi(t')\xi(t'')\right\rangle \\ &= 2D \int_0^t dt' \int_0^t dt'' \delta(t'-t'') \\ &= 2Dt \end{aligned} \tag{12}$$

So

$$\partial_t P(x,t) = -\partial_x\left((-mg)P(x,t)\right) + D\partial_x^2 P(x,t) \tag{13}$$

No stationary solution for $-\infty < x < \infty$ but for $x > 0$ there is

$$P_{st}(x) = \text{const} \times e^{-\frac{g}{\gamma D}x} \tag{14}$$

For $D = \frac{kT}{m\gamma}$ get barometric equation

$$P_{st}(x) = \text{const} \times e^{-\frac{mgx}{kT}} \tag{15}$$

which is the barometric equation.

2.3 Langevin dynamics: Brownian particles with a linear drift

$x(t)$ follows a stochastic process of the form

$$\dot{x}(t) = ax(t) + b\xi(t) \tag{16}$$

with $\xi(t)$ a Gaussian random variable with $\langle \xi(t)\rangle = 0$ and $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. Since $\xi(t)$ is a random variable, $x(t)$ will be random too.

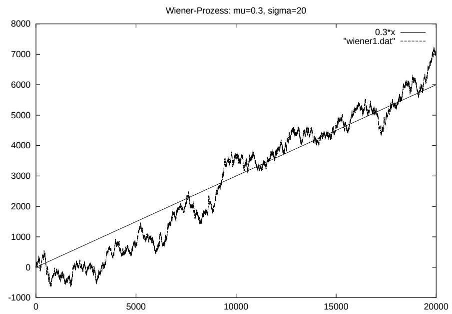


Figure 2: Brownian motion with constant drift ($dx = adt + bdW$) $a = 0.3$, $b = 20$

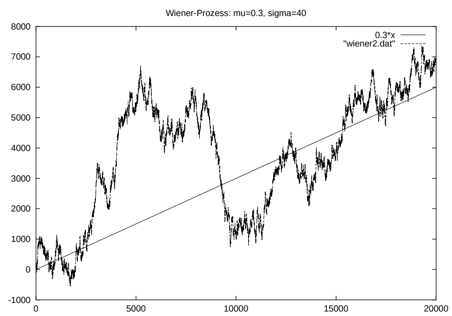


Figure 3: Brownian motion with constant drift ($dx = adt + bdW$) $a = 0.3$, $b = 40$

Let us assume we can integrate as in determinstic equations. Then one gets

$$x(t) = x(0)e^{at} + be^{at} \int_0^t dt' e^{-at'} \xi(t') dt' \tag{17}$$

Exercise: Check by differentiation that this solves eq. (16).

From this one gets

$$\langle x(t) \rangle = x(0)e^{at} + be^{at} \int_0^t dt' e^{-at'} \underbrace{\langle \xi(t') \rangle}_{=0} dt' = x(0)e^{at} \tag{18}$$

and

$$\begin{aligned}
\left\langle x^2(t) \right\rangle &= x(0)^2 e^{2at} + 2x(0)be^{2at} \int_0^t dt' e^{-at'} \underbrace{\langle \xi(t') \rangle}_{=0} + b^2 e^{2at} \int_0^t dt' \int_0^t dt'' e^{-at'} e^{-at''} \underbrace{\langle \xi(t)\xi(t') \rangle}_{=\delta(t-t')} \\
&= x(0)^2 e^{2at} + b^2 e^{2at} \int_0^t dt' e^{-2at'} \\
&= x(0)^2 e^{2at} + \frac{b^2}{2a} \left(e^{2at} - 1 \right) \qquad (19)
\end{aligned}$$

The variance is therefore

$$\left\langle x^2(t)\right\rangle - \langle x(t)\rangle^2 = \frac{b^2}{2a}\left(e^{2at}-1\right) \tag{20}$$

Note that for small drifts a we recover

$$\begin{aligned}\left\langle x^2(t)\right\rangle - \langle x(t)\rangle^2 &= \frac{b^2}{2a}\left(1+2at+\mathcal{O}(a^2)-1\right)\\ &= b^2 t + \mathcal{O}(a) \end{aligned} \tag{21}$$

I.e. for $a \to 0$ (no drift) one has $\langle x^2\rangle - \langle x\rangle^2 = b^2 t$, the result known from the driftless diffusion equation.

So we have seen that

$$\dot{x}(t) = ax(t) + b\xi(t) \tag{22}$$

leads to a Gaussian distribution of $x(t)$ with

$$\begin{aligned}\langle x(t)\rangle &= x(0)e^{at}\\ \left\langle x^2(t)\right\rangle - \langle x(t)\rangle^2 &= \frac{b^2}{2a}\left(e^{2at}-1\right)\end{aligned} \tag{23}$$

The corresponding differential equation describing $P(x,t)$ is

$$\partial_t P(x,t) = -\partial_x\left(axP(x,t)\right) + \frac{b^2}{2}\partial_x^2 P(x,t) \tag{24}$$

as one can check by direct inspection. The corresponding stochastic process (22) is known as an **Ornstein-Uhlenbeck process**.

2.4 General rule

We have seen in several examples that a process

$$\dot{x}(t) = f(x)x(t) + \xi(t) \tag{25}$$

with $\langle\xi(t)\xi(t')\rangle = 2D$ leads to a Fokker-Planck equation of the form

$$\partial_t P(x,t) = -\partial_x\left(P(x,t)f(x)\right) + D\partial_x^2 P(x,t) \tag{26}$$

- $f(x)$ is the deterministic force acting on the particle, hence the first term on the right-hand-side is called the deterministic term.
- the second term (proportional to D) comes from the randomness, it is referred to as the diffusion term

2.5 Langevin dynamics in statistical mechanics

Again assume overdamped dynamics of the form

$$\dot{\phi}(t) = -\frac{\partial H}{\partial \phi} + \xi(t) \tag{27}$$

with $\langle \xi(t)\xi(t')\rangle = 2kT$ (and $\langle xi(t)\rangle = 0$). The the corresponding Fokker-Planck equation turns out as

$$\partial_t P(\phi, t) = -\partial_\phi \left(P(\phi, t)F(\phi)\right) + kT\partial_\phi^2 P(\phi, t) \tag{28}$$

with the force $F(\phi) = -\partial_\phi H(\phi)$.

Question: what is the stationary distribution of this ?

You may check that inserting $P(\phi) = \text{const} \times exp\left(-\frac{H(\phi)}{kT}\right)$ into the right-hand-side of (28) gives $\partial_t P(\phi) = 0$, i.e. the Boltzmann distribution is the stationary solution of this Fokker-Planck equation.

2.6 How to put a stochastic differential equation onto a computer ?

Assume you want to solve

$$\dot{x}(t) = F(x(t)) + \sqrt{2T}\xi(t) \tag{29}$$

(with some force field $F(x)$, and $\langle \xi(t)^2\rangle = 1$) on a computer ? Normally you would discretise as follows:

$$x(t+\Delta) = x(t) + \Delta[F(x(t)) + \sqrt{2T}\xi(t)]. \tag{30}$$

If you do this, you would get the following iteration for the probability distribution $P(x,t)$ of x at time t:

$$\begin{aligned} P(x,t+\Delta) - P(x,t) &= \langle \delta(x - x(t+\Delta)) \rangle - \langle \delta(x - x(t)) \rangle \\ &= \left\langle \delta(x - x(t) - \Delta F(x(t) - \Delta\sqrt{2T}\xi(t)) \right\rangle - \langle \delta(x - x(t)) \rangle \\ &= -\partial_x \left\langle \delta(x - x(t)) \big[\Delta F(x(t) - \Delta\sqrt{2T}\xi(t))\big] \right\rangle \end{aligned} \tag{31}$$

I.e.

$$\begin{aligned} \frac{P(x,t+\Delta) - P(x,t)}{\Delta} &= -\partial_x \left[F(x) \langle \delta(x - x(t)) \rangle\right] + \mathcal{O}(\Delta) \\ &= -\partial_x \left[F(x) P(x,t)\right] + \mathcal{O}(\Delta), \end{aligned} \tag{32}$$

so the diffusion term in the Fokker-Planck equation is missing.

If instead you use

$$x(t+\Delta) = x(t) + \Delta F(x(t)) + \sqrt{\Delta}\sqrt{2T}\xi(t) \tag{33}$$

then you get

$$\begin{aligned} P(x,t+\Delta) - P(x,t) &= \langle \delta(x - x(t+\Delta)) \rangle - \langle \delta(x - x(t)) \rangle \\ &= \left\langle \delta(x - x(t) - \Delta F(x(t) - \sqrt{\Delta}\sqrt{2T}\xi(t)) \right\rangle - \langle \delta(x - x(t)) \rangle \\ &= -\partial_x \left\langle \delta(x - x(t)) \big[\Delta F(x(t) + \sqrt{\Delta}\sqrt{2T}\xi(t))\big] \right\rangle \\ &\quad + \frac{1}{2}\Delta(2T)\partial_x^2 \left\langle \delta(x - x(t))\xi(t)^2 \right\rangle \end{aligned} \tag{34}$$

I.e.

$$\frac{P(x,t+\Delta) - P(x,t)}{\Delta} = -\partial_x \left[F(x)P(x,t)\right] + T\partial_x^2 P(x,t) + \mathcal{O}(\Delta) \tag{35}$$

so the diffusion term is restored.

3 Markov processes

Markov processes are processes for which the probability density for x_n at time t_n at previous history $\{(x_i, t_i)|1 \leq i \leq n-1\}$ with $t_1 < t_2 < \cdots < t_{n-1} < t_n$ depends on (x_n, t_n) only, i.e

$$p(x_n, t_n|x_{n-1}, t_{n-1}; \cdots; x_1, t_1) = p(x_n, t_n|x_{n-1}, t_{n-1}). \tag{36}$$

For times $t_1 < t_2 < t_3$ this means for example that

$$p(x_3, t_3; x_2, t_2; x_1, t_1) = p(x_3, t_3|x_2, t_2)p(x_2, t_2|x_1, t_1)p(x_1, t_1) \tag{37}$$

3.1 Chapman-Kolmogorov equation

Now integrate (37) over x_2 and divide by $p(x_1, t_1)$, and use the standard rule for conditional probabilities $p(x_3, t_3|x_1, t_1) = p(x_3, t_3; x_1, t_1)/p(x_1, t_1)$. Get

$$p(x_3, t_3|x_1, t_1) = \int dx_2 \; p(x_3, t_3|x_2, t_2)p(x_2, t_2|x_1, t_1). \tag{38}$$

This is the Chapman-Kolmogorov equation, it is a non-linear deterministic integral equation.

3.2 Master equation

Now go to continuous time and set $t_3 = t_2 + \tau$ and consider

$$p(x_3, t_2 + \tau|x_1, t_1) = \int dx_2 p(x_3, t_2 + \tau|x_2, t_2)p(x_2, t_2|x_1, t_1) \tag{39}$$

Now we have

$$p(x_3, t_2 + \tau|x_2, t_2) = \frac{p(x_3, t_2 + \tau|x_2, t_2)}{\int dx' p(x', t_2 + \tau|x_2, t_2)} \tag{40}$$

since the denominator $N = \int dx' p(x', t_2 + \tau | x_2, t_2)$ is equal to one. Now expand in terms of τ

$$\begin{aligned} p(x_3, t_2 + \tau | x_2, t_2) &= \frac{p(x_3, t_2 + \tau | x_2, t_2)}{\int dx' p(x', t_2 + \tau | x_2, t_2)} \\ &= \underbrace{p(x_3, t_2 | x_2, t_2)}_{=\delta(x_3 - x_2)} \\ &\quad + \frac{\tau}{N} W_{t_2}(x_3 | x_2) - \frac{\tau}{N^2} \underbrace{p(x_3, t_2 | x_2, t_2)}_{=\delta(x_3 - x_2)} \int dx' W_{t_2}(x' | x_2) + \mathcal{O}(\tau^2). \end{aligned} \tag{41}$$

Here we have introduced the transition rates

$$W_t(x'|x) = \left. \frac{\partial p(x', t + \tau | x, t)}{\partial_\tau} \right|_{\tau = 0} \tag{42}$$

from x to x' at time t. So we find

$$p(x_3, t_2 + \tau | x_2, t_2) = \delta(x_3 - x_2) + \tau \left(W_{t_2} - \int dx' W_{t_2}(x'|x_2) \delta(x_3 - x_2) \right) \tag{43}$$

Insert this into (39). Get

$$\begin{aligned} p(x_3, t_2 + \tau | x_1, t_1) &= p(x_3, t_2 | x_1, t_1) \\ &\quad - \tau \int dx' W_{t_2}(x'|x_3) p(x_3, t_2 | x_1, t_1) \\ &\quad + \tau \int dx_2 W_{t_2}(x_3 | x_2) p(x_2, t_2 | x_1, t_1) \end{aligned} \tag{44}$$

From this one finds

$$\partial_{t_2} p(x_3, t_2 | x_1, t_1) = \int dx_2 \left(\underbrace{W_{t_2}(x_3|x_2) p(x_2, t_2 | x_1, t_1)}_{gain} - \underbrace{W_{t_2}(x_2|x_3) p(x_3, t_2 | x_1, t_1)}_{loss} \right) \tag{45}$$

This is the Master equation.

3.3 Fokker-Planck equation

$W_t(x'|x)$ is the transition rate from x to x' at time t. If say $x' = x + \Delta x$ then it will be more convenient to express this in terms of probabilities to jump/move by Δx

given the particle is at x at t. We will adopt the notation $\widetilde{W}_t(\delta x, x) = W_t(x + \Delta x|x)$ in following, and drop the tilde. The Master equation then reads

$$\begin{aligned}
\partial_t p(x,t|x_0,t_0) &= \int d(\Delta x) W_t(\Delta x, x-\Delta x) p(x-\Delta x, t|x_0,t_0) \\
&\quad - p(x,t|x_0,t_0) \underbrace{\int d(\Delta x) W_t(-\Delta x, x)}_{=1}
\end{aligned} \tag{46}$$

Now use

$$W_t(\Delta x, x-\Delta x) p(x-\Delta x, t|x_0,t_0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\Delta x)^n \frac{\partial^n}{\partial x^n} \left[W_t(\Delta x, x) p(x,t|x_0,t_0) \right] \tag{47}$$

Inserting this one obtains

$$\partial_t p(x,t|x_0,t_0) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left[a_n(x,t) p(x,t|x_0,t_0) \right] \tag{48}$$

with

$$a_n(x,t) = \int d(\Delta x) (\Delta x)^n W_t(\Delta x, x) \tag{49}$$

This is referred to as the **Kramers-Moyal expansion**.

Truncating the series after two terms ('only small jumps during the transitions') leads to the Fokker-Planck equation

$$\partial_t p(x,t|x_0,t_0) = -\partial_x \big(a_1(x,t) P(x,t) \big) + \frac{1}{2} \partial_x^2 \big(a_2(x,t) P(x,t) \big) \tag{50}$$

with drift and diffusion terms

$$\begin{aligned}
a_1(x,t) &= \int d(\Delta x) (\Delta x) W_t(\Delta x, x) \\
a_2(x,t) &= \int d(\Delta x) (\Delta x)^2 W_t(\Delta x, x)
\end{aligned} \tag{51}$$

4 Stochastic differential equations

4.1 $dW^2 = dt$

Consider again the standard Brownian motion

$$\dot{x}(t) = \xi(t) \tag{52}$$

with $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. One defines the standard Wiener process by

$$W(t) = w(0) + x(t) - x(0) = w(0) + \int_0^t dt' \dot{x}(t') = w(0) + \int_0^t dt' \xi(t'), \tag{53}$$

which is often also written as

$$\frac{dW(t)}{dt} = \xi(t) \tag{54}$$

or

$$dW(t) = \xi(t) dt. \tag{55}$$

In particular one has

$$\langle W(t)\rangle = w(0), \qquad \left\langle (W(t) - w(0))^2 \right\rangle = t. \tag{56}$$

Also

$$\left\langle (W(t+\tau) - W(t))^2 \right\rangle = \int_t^{t+\tau} dt' \int_t^{t+\tau} dt'' \left\langle \xi(t')\xi(t'')\right\rangle = \tau \tag{57}$$

i.e. for small τ have

$$\left\langle (dW(t))^2 \right\rangle = \left\langle (W(t+dt) - W(t))^2 \right\rangle = dt \tag{58}$$

As a short-hand one writes

$$dW^2 = dt \tag{59}$$

One can show that this holds not only as an average, but that $(dW)^2$ may be substituted by dt in all operations of stochastic calculus.

4.2 Ito's Lemma

4.2.1 Example

Consider

$$W(t)^2 - W(0)^2 = \int_0^t d(W^2) \tag{60}$$

Let us set the starting point to $W(0) = 0$. We know

$$dW = \xi dt \tag{61}$$

but how can one compute $\int_0^t d(W^2)$ from this ?

Naively one would do as follows

$$d(W^2) = 2W dW \tag{62}$$

i.e.

$$\begin{aligned}
\int_0^t d(W^2) &= 2\int_0^t W(t')dW(t') \\
&= 2\int_0^t dt' W(t')\xi(t')dt' \\
&\approx 2\sum_{i=0}^{n} W\left(\frac{i}{n}t\right)\left(W\left(\frac{i+1}{n}t\right) - W\left(\frac{i}{n}t\right)\right)
\end{aligned} \tag{63}$$

Now

$$\left(W\left(\frac{i+1}{n}t\right) - W\left(\frac{i}{n}t\right)\right) \tag{64}$$

is the increment of a Wiener process, and hence independent of $W\left(\frac{i}{n}t\right)$. It follows

$$\begin{aligned}
\langle W(t)^2 \rangle &= 2\sum_{i=0}^{n} \underbrace{\left\langle W\left(\frac{i}{n}t\right)\right\rangle}_{=0} \left\langle \left(W\left(\frac{i+1}{n}t\right) - W\left(\frac{i}{n}t\right)\right)\right\rangle \\
&= 0
\end{aligned} \tag{65}$$

This does not make sense, as we know that $\langle W(t)^2 \rangle = t$.

What went wrong?

What we did is expand

$$d(W^2) = 2W dW + \mathcal{O}(dW^2) \tag{66}$$

and ignored higher order terms, including dW^2.

But $dW^2 = dt$ so need to include second order terms. Then one gets

$$d(W^2) = 2W dW + dW^2 = 2W dW + dt \tag{67}$$

I.e.

$$\begin{aligned}
\left\langle W^2(t) \right\rangle &= \left\langle \int_0^t d(W^2(t')) \right\rangle \\
&= 2 \underbrace{\left\langle \int_0^t W(t') dW(t') \right\rangle}_{=0} + \int_0^t dt' \\
&= t
\end{aligned} \tag{68}$$

which is the correct result.

4.2.2 General case

Assume $x(t)$ obeys the Ito-SDE

$$dx(t) = a[x(t), t]dt + b[x(t), t]dW(t) \tag{69}$$

Now take a function $y(t) = f[x(t)]$, what SDE does $y(t)$ obey ? Use previous results to expand $df[x(t)]$:

$$\begin{aligned}
df[x(t)] &= f[x(t) + dx(t)] - f[x(t)] \\
&= f'[x(t)]dx(t) + \frac{1}{2} f''[x(t)]dx(t)^2 + \ldots \\
&= f'[x(t)] \left(a[x(t), t]dt + b[x(t), t]dW(t) \right) + \frac{1}{2} f''[x(t)] b[x(t), t]^2 (dW(t))^2 + \mathcal{O}(dt^2)
\end{aligned} \tag{70}$$

Now we use that $dW(t)^2 = dt$ and obtain

$$df[x(t)] = \left(f'[x(t)]a[x(t),t] + \underbrace{\frac{1}{2} f''[x(t)]b[x(t),t]^2}_{\text{'Ito term'}} \right) dt + b[x(t),t]f'[x(t)]dW(t) \quad (71)$$

Note that the term containing $f''[x(t)]$ would not be present in normal calculus
So the recipe for changes of variables in the Ito-formalism is:

1. when performing a Taylor expansion, terms of second order in dW have to be taken into account

2. use: $dW(t)^2 = dt$

4.3 General SDEs

Consider a general (ordinary) differential equation of the form

$$\dot{x}(t) = a[x(t),t] + b[x(t),t]\xi(t) \quad (72)$$

with a time- and $x-$ dependent drift $a(x,t)$ and a time- and $x-$dependent magnitude of the fluctuations $b[x,t]$.
This can be seen as an equivalent formulation of the following integral equation (which we get from (72) by integration over t):

$$x(t) - x(t_0) = \int_{t_0}^{t} dt' A[x(t'),t'] + \int_{t_0}^{t} dt' B[x(t'),t']\xi(t') \quad (73)$$

So the question is: what do we mean by a stochastic integral of the form

$$\int_{t_0}^{t} dt' f(t')\xi(t') \quad (74)$$

with $f(t)$ a deterministic or a random function ?

5 Stochastic integration - Ito Calculus

5.1 General remarks (section not proof read)

Want to define

$$\int_{t_0}^{t} dt' f(t')\xi(t'). \tag{75}$$

Let us write $\xi(t') = \frac{dW(t')}{dt'}$, then $W(t')$ will also be a random function and we have

$$\int_{t_0}^{t} dt' f(t')\xi(t') = \int_{t_0}^{t} dt' f(t') \frac{dW(t')}{dt'} = \int_{t_0}^{t} f(t') dW(t'). \tag{76}$$

In order to obtain an interpretation in terms of Riemannian sums, two standard choices are used:

1. **Ito interpretation**

$$I \int_{t_0}^{t} f(t') dW(t') = \lim_{N \to \infty} \sum_{i=1}^{N} f(x(t_{i-1})) \left(W(t_i) - W(t_{i-1})\right) \tag{77}$$

2. **Stratonovich interpretation**

$$S \int_{t_0}^{t} f(t') dW(t') = \lim_{N \to \infty} \sum_{i=1}^{N} f\left(\frac{x(t_i) + x(t_{i-1})}{2}\right) \left(W(t_i) - W(t_{i-1})\right) \tag{78}$$

Note that the two interpretations are not equivalent. They differ in the argument at which $f(\cdot)$ is to be taken in the summation, and for the same integral they will in general lead to different expressions.

Example: $\int_{t_0}^{t} dt' W(t') dW(t')$

ITO CALCULATION: Write $t_i = t_0 + \frac{i}{N}(t - t_0)$.

$$\begin{aligned}
I\int_{t_0}^{t} dt' W(t')dW(t') &= \lim_{N\to\infty}\sum_{i=1}^{N} W(t_{i-1})\left(W(t_i) - W(t_{i-1})\right) \\
&= \lim_{N\to\infty}\sum_{i=1}^{N} W_{i-1}\Delta W_i \\
&= \lim_{N\to\infty}\frac{1}{2}\sum_{i=1}^{N}\left[(W_{i-1}+\Delta W_i)^2 - (W_{i-1})^2 - (\Delta W_i)^2\right] \\
&= \frac{1}{2}\left[W(t)^2 - W(t_0)^2\right] - \frac{1}{2}\sum_{i=1}^{N}(\Delta W_i)^2 \\
&= \frac{1}{2}\left[W(t)^2 - W(t_0)^2\right] - \frac{1}{2}(t - t_0) \qquad (79)
\end{aligned}$$

STRATONOVICH CALCULATION:

$$\begin{aligned}
S\int_{t_0}^{t} dt' W(t')dW(t') &= \lim_{N\to\infty}\sum_{i=1}^{N}\frac{W_{i-1}+W_i}{2}\left(W_i - W(i-1)\right) \\
&= \lim_{N\to\infty}\frac{1}{2}\sum_{i=1}^{N}\left[W_i^2 - W_{i-1}^2\right] \\
&= \frac{1}{2}\left[W^2(t) - W^2(t_0)\right] \qquad (80)
\end{aligned}$$

So we realise that

$$\begin{aligned}
\text{Ito}\int_{t_0}^{t} dt' W(t')dW(t') &= \frac{1}{2}\left[W(t)^2 - W(t_0)^2\right] - \frac{1}{2}(t - t_0) \qquad (81) \\
\text{Stratonovich}\int_{t_0}^{t} dt' W(t')dW(t') &= \frac{1}{2}\left[W(t)^2 - W(t_0)^2\right] \qquad (82)
\end{aligned}$$

Think back of section 4.2.1. Consider

$$W(t)^2 - W(0)^2 = \int_0^t d(W^2) \qquad (83)$$

and set $W(0)$=0. We know $\langle W(t)^2\rangle = t$

Naively setting $d(W^2) = 2WdW$ gives

$$W(t)^2 = 2\int_0^t WdW \qquad (84)$$

STRATONOVICH:

If this integral is interpreted in the Stratonovich sense, everything is file

$$W(t)^2 = 2S\int_0^t WdW = W(t)^2. \tag{85}$$

So applying the chain rule of usual calculus ($d(W^2) = 2WdW$) seems fine for Stratonovich integrals. ITO:

Here we have to use the Ito-rule

$$d(W^2) = 2WdW + 2(dW)^2 = 2WdW + dt \tag{86}$$

and get

$$\begin{aligned} W(t)^2 &= I\int_0^t (dW)^2 = W(t)^2 \\ &= 2\underbrace{I\int_0^t WdW}_{=\frac{1}{2}W(t)^2-\frac{1}{2}t=0} + \int_0^t dt' \\ &= W(t)^2 \end{aligned} \tag{87}$$

Take home message:

If stochastic integrals are interpreted in the Stratonovich sense, the normal rules of calculus may be applied. If the Ito-interpretation is used, $dW^2 = dt$ must be taken into account, and Ito's lemma is to be used for transformation of variables.

5.2 Ito versus Stratonovich SDE (section not proof read)

Take general SDE

$$dx(t) = a[x(t,t]dt + b[x(t),t]dW. \tag{88}$$

This will lead to two different processes $x(t)$ depending on whether you integrate this according the Ito- or Stratonovich prescription.

What do the corresponding Fokker-Planck equations look like in the Ito and Stratonovich interpretation, respectively ?

Have

$$x(t+dt) - x(t) \;\; = \;\; a[x(t),t]dt + b[x(t_\alpha),t_\alpha](W(t+dt) - W(t)) \qquad (89)$$

with $t_\alpha = t + \alpha dt$, $\alpha = 0$ for Ito, and $\alpha = 1/2$ for Stratonovich. Now have

$$\begin{aligned} b[x(t_\alpha),t_\alpha] &= b[x(t),t] + \partial_x b[x(t),t](x(t_\alpha) - x(t)) + \ldots \\ &= b[x(t),t] + \partial_x b[x(t),t]\big\{a[x(t),t]\alpha dt + b[x(t),t][W(t+\alpha dt) - W(t)]\big\} \end{aligned} \qquad (90)$$

So get

$$\begin{aligned} x(t+dt) - x(t) &= a[x(t),t]dt + b[x(t_\alpha),t_\alpha](W(t+dt) - W(t)) \\ &= a[x(t),t]dt + b[x(t),t](W(t+dt) - W(t)) \\ &\quad + \partial_x b[x(t),t]\Big\{a[x(t),t]\alpha dt(W(t+dt) - W(t))\Big\} \\ &\quad + \partial_x b[x(t),t]\Big\{b[x(t),t][W(t+\alpha dt) - W(t)]\,[W(t+dt) - W(t)]\Big\} \end{aligned} \qquad (91)$$

Now taking averages gives

$$\begin{aligned} \langle x(t+dt) - x(t)\rangle &= a[x(t),t]dt + b[x(t),t]\,\langle [W(t+\alpha dt) - W(t)][W(t+dt) - W(t)]\rangle \\ &= a[x(t),t]dt + b[x(t),t]\int_t^{t+dt} dt' \int_t^{t+\alpha dt} dt''\,\langle \xi(t')\xi(t'')\rangle \\ &= a[x(t),t]dt + b[x(t),t]\alpha dt \end{aligned} \qquad (92)$$

So the drift of $x(t)$ is given by by $a[x(t),t]$ in the Ito understanding of the SDE ($\alpha = 0$) and by $a[x(t),t] + \frac{1}{2}b[x(t),t]$ in the Stratonovich interpretation ($\alpha = 1/2$). A similar calculation for the diffusion term of the Fokker-Planck eq. gives $\frac{1}{2}b[x(t),t]^2$ in both cases.

Conclusion: Integrating the SDE

$$dx(t) = a[x(t,t]dt + b[x(t),t]dW \tag{93}$$

in the Ito sense leads to a FP equation of the form

$$\partial_t P(x,t) = \partial_x (a[x(t),t]P(x,t)) + \frac{1}{2}\partial_x^2 \left(b[x(t),t]^2 P(x,t)\right). \tag{94}$$

Integrating the same SDE

$$dx(t) = a[x(t,t]dt + b[x(t),t]dW \tag{95}$$

in the Stratonovich sense leads to a FP equation of the form

$$\partial_t P(x,t) = \partial_x \left(a[x(t),t] + \frac{1}{2}b[x(t),t]\partial_x b[x(t),t]\right) P(x,t)) + \frac{1}{2}\partial_x^2 \left(b[x(t),t]^2 P(x,t)\right). \tag{96}$$

Ito	Stratonovich
$(dW)^2 = dt$	$(dW)^2 = dt$
$\langle W dW \rangle = 0$	$2\langle W dW \rangle = dt$
$d(f(W,t)) = \frac{\partial f}{\partial W}dW + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}dt$	$d(f(W,t)) = \frac{\partial f}{\partial W}dW + \frac{\partial f}{\partial t}dt$
$dx = adt + bdW$ corresponds to FP-eq $\partial_t = \partial_x(aP) + \frac{1}{2}\partial_x^2(b^2P)$	$dx = adt + bdW$ corresponds to FP-eq $\partial_t = \partial_x(aP + \frac{1}{2}b\partial_x b) + \frac{1}{2}\partial_x^2(b^2P)$
functions evaluated before the increments	functions are evaluated simultenously to increments
more common in finance	more common in physics

See Bouchaud/Potters and Gardiner books.

6 Financial derivatives

6.1 General remarks

basic goods

- goods, commodities, i.e. oil, gold, cars ...

- stocks
- currency
- bonds

Since the opening of the Chicago Board 1973 also with derived goods, so-called financial derivatives.
Assume producer of cars needs 1000 tons of iron, not now, but in 2 months from now. He could wait the two months, and then buy at the price of that day. There is some risk associated with this, as the price in two months from now is uncertain.
Concept of *futures* and *forwards*: the car-producer signs a contract with a supplier of iron in advance: this contract is agreed on now, and the buying/selling of the iron is then made in two months from now.
Will here consider mostly so-called *options*:

European call option: At a prescribed time in the future (expiry date) the holder of an option may (but does not have to) purchase a prescribed amount of a prescribed asset (the so-called underlying asset) at a previously agreed price (the strike price).The contract is a right to buy, but not an obligation to buy. The seller (the so-called writer of the option) on the other hand has a potential obligation to sell, if the holder chooses to buy.

Questions for financial practitioners: How much would you be willing to pay for such a right, i.e. what is the value of such an option ? How can the writer minimise the risk associated with his potential obligation to sell ?

6.2 Options

Example: Today is 5 May 2006. How much is the following option worth: On 5 August 2006 the holder of the option may (if he so wishes) purchase 1 ton of iron at a price of 100 dollars per ton.

- Assume the price of iron is at 120 dollars on 5 August. Then the option is worth 20 dollars (as the holder will choose to use the option, buy 1 ton for 100 dollars, and sell for 120 dollars).

- Assume the price of iron is at 90 dollars on 5 August, then it would not be sensible to use the option. Using the option would mean to buy at a price of 100 dollars, when we can buy elsewhere for 90 dollars. The option is worthless.

Of course the price of an option has to be determined at the time the contract is signed (i.e. 5 May 2006 in our case). The price of iron in August is not known at this time, only stochastic statements can be made. Assume for example we had a model which told us the price of iron will be 120 dollars on 5 August with probability p and 90 dollars with probability $1-p$. Then the expected value of the option is $p \times 20$ dollars.

General: If T is the expiry time of a European Call option and E the strike price, then the value of the option is given by

$$G = \max(S(T) - E, 0) \tag{97}$$

where $S(T)$ is the price of the underlying at T. What one needs is

- a good model which makes stochastic statements regarding $S(T)$, where T is some time in the future. If for example we had the probability distribution of $S(T)$ we could perform an average, and compute the expected fair price.

- stochastic calculus to do the computation

6.3 No free lunch

One of the most important concepts underlying the pricing of financial derivatives is that of arbitrage. Roughly speaking this means that there is no such thing as a free lunch. Arbitrage means the possibility to make money at zero risk. Assume a

stock is at 100 dollar in New York, and at 92 Euro in Frankfurt. The exchange rate is $93 Euro/dollar$. Then there is the following arbitrage opportunity:

- buy 100 stocks in Frankfurt (pay 9200 Euros)
- take them to New York, sell them there (get 10000 dollar)
- change the 10000 dollar for 9300 Euro
- zero risk, but gaines 100 Euro

In a sufficiently transparent and efficient market such a situation can never occur. Everybody would be doing the above operation, and the price in Frankfurt would go up (everybody buys there), and the one in New York would go down (everybody sells there), until the arbitrage opportunity is levelled out.

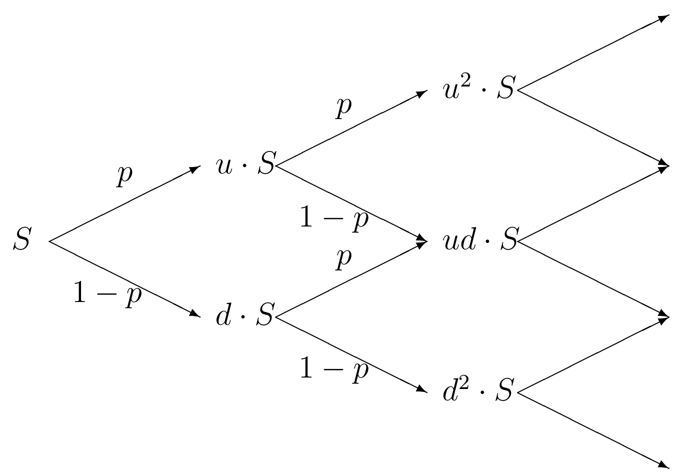
The pricing mechanisms in the following sections are based on the 'no-arbitrage principle'.

7 Pricing of financial derivatives, the Black-Scholes formula

7.1 The Cox-Ross-Rubinstein-Modell

We here consider a simple model of a random price process $S(t)$ in the timer interval $[0, T]$. We discretise time $t_k = \frac{k}{n}T$ and assume that the price moes up or down at each time step stochastically according to

- the price goes up by a factor of $u < 1$ with probability p, i.e. $S_{k+1} = uS_k$
- the price goes down by a factor of $d < 1$ with probability $1-p$ i.e. $S_{k+1} = dS_k$



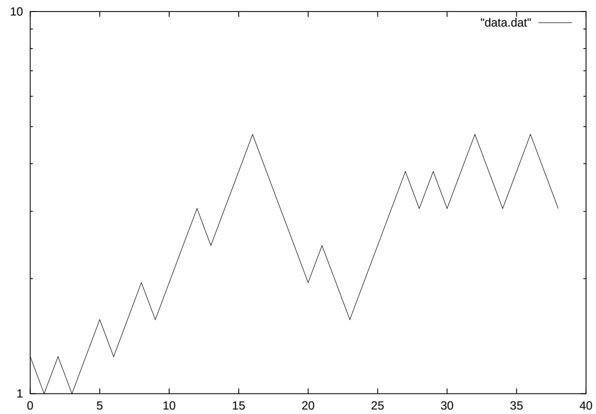
u, d, p are time independent model parameters, and we assume the price movement at step $k+1$ is independent of all previous time steps.
A possible trajectory (price time series) looks like this
Now note that $\log S_{k+1} = \log S_k + \log u$ if the price goes up, and $\log S_{k+1} = \log S_k + \log d$ if the price goes down. The increments of $\log S$ are thus random

$$\log S_{k+1} = \log S_k + \xi \tag{98}$$

and in the continuous time limit one can show (central limit theorem) that $logS(t)$ follows a random walk with drift

$$\frac{d}{dt} \log S(t) = \lambda + \sigma_0 \xi(t) \tag{99}$$

with drift parameter λ and volatility σ_0.



log $S(t)$ is thus a Gaussian random variable, and $S(t)$ is said to follow a 'log-normal' distribution (its log is Gaussian).

7.2 Black-Scholes model

Consider the following goods:

- a bond with interest rate r
- a stock with price process $S(t)$
- a European call option on this stock

We assume

- $S(t)$ follows a log-normal random walk
- the volatility σ of this process is known
- the interest rate r is known and constant
- no dividends on the stock
- no arbitrage
- short selling is allowed: one may sell stocks that one does not own

- trading is in continuous portions
- no transaction costs

Denote the value of the option by $G(t)$. Both $S(t)$ and $G(t)$ are stochastic processes. The value $B(t)$ of the bond is determinstic, one has

$$B(t) = B(0)e^{rt} \tag{100}$$

i.e. $dB(t) = rB(t)dt$. Construct a riskless portfolio $P(t)$ (consisting of stocks and bonds). If $P(t)$ is riskless (i.e. deterministic), the no arbitrage principle dictates that $dP(t) = rP(t)dt$. From this try to obtain the process for $G(t)$.

7.3 The Black-Scholes differential equation

Stock price follows a stochastic process of the form

$$dS = \sigma_0 \, Sdz + \mu_0 \, Sdt. \tag{101}$$

The value of the option at time t will depend on $S(t)$ and t, i.e. $G(t) = G(S,t)$. Ito's lemma gives

$$\begin{aligned} dG &= \frac{\partial G}{\partial S} dS + \frac{1}{2}\frac{\partial^2 G}{\partial G^2}(dS)^2 + \frac{\partial G}{\partial t} dt \\ &= \sigma_0 S \frac{\partial G}{\partial S} dz + \left(\mu_0 S \frac{\partial G}{\partial S} + \frac{1}{2}\sigma_0^2 S^2 \frac{\partial^2 G}{\partial S^2} + \frac{\partial G}{\partial t} \right) dt \end{aligned} \tag{102}$$

Now construction of the riskless portfolio. Assume it consists of a number Δ of stocks and a short position of one option. Value of the portfolio

$$P(t) = \Delta \times S(t) - G(t) \tag{103}$$

Since

$$d(\Delta \times S)(t) = \Delta\sigma_0 \, Sdz + \Delta\mu_0 \, Sdt, \tag{104}$$

one finds

$$\begin{aligned} dP(t) &= d(\Delta S(t)) - dG(t) \\ &= d(\Delta S) - \sigma_0 S\frac{\partial G}{\partial S}dz - \left(\mu_0 S\frac{\partial G}{\partial S} + \frac{1}{2}\sigma_0^2 S^2\frac{\partial^2 G}{\partial S^2} + \frac{\partial G}{\partial t}\right)dt \\ &= -\sigma_0 S\left(\frac{\partial G}{\partial S} - \Delta\right)dz - \left(\mu_0 S\left(\frac{\partial G}{\partial S} - \Delta\right) + \frac{1}{2}\sigma_0^2 S^2\frac{\partial^2 G}{\partial S^2} + \frac{\partial G}{\partial t}\right)dt. \end{aligned} \tag{105}$$

The choice

$$\Delta = \frac{\partial G}{\partial S} \tag{106}$$

thus eliminates the random term (proportional to dz):

$$dP = \left(\frac{1}{2}\sigma_0^2 S^2\frac{\partial^2 G}{\partial S^2} + \frac{\partial G}{\partial t}\right)dt. \tag{107}$$

If $\Delta(t)$ is chosen according to this presciption the value of the portfolio becomes determinstic. This is referred to as *Delta hedging.* Using the no-arbitrage principle one concludes that $P(t)$ must grow exponentially with growth rate equal to the interest rate r

$$dP(t) = r\,P(t)dt \tag{108}$$

Now use

$$P(t) = \frac{\partial G}{\partial S}S(t) - G(t) \tag{109}$$

(insert (106) in (103)):

$$\left(\frac{1}{2}\sigma_0^2 S^2\frac{\partial^2 G}{\partial S^2} + \frac{\partial G}{\partial t}\right)dt = r\left(-\frac{\partial G}{\partial S}S(t) + G(t)\right)dt \tag{110}$$

This gives the **Black-Scholes differential equation**:

$$\frac{\partial G}{\partial t} + rS(t)\frac{\partial G}{\partial S} + \frac{1}{2}\sigma_0^2 S^2\frac{\partial^2 G}{\partial S^2} - rG(t) = 0 \tag{111}$$

7.4 Solution of the BS differential equation

Now have to solve BS equation subject to the boundary condition at the expiry date

$$G(S,T) = max(S(T) - E, 0) \tag{112}$$

T is the expiry date and E the strike price

7.4.1 Transformation of variables

Introduce x, τ and $v(x,t)$ according to

$$\begin{aligned} S &= Ee^x && (113)\\ t &= T - \frac{2\tau}{\sigma^2} && (114)\\ G &= Ev(x,\tau). && (115) \end{aligned}$$

This turns the BD eq. into the (generalised) diffusion equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (\gamma - 1)\frac{\partial v}{\partial x} - \gamma v. \tag{116}$$

with a dimensionless quantity $\gamma = \frac{2r}{\sigma_0^2}$ dimensionslos. The boundary condition becomes

$$v(x,0) = max\,(e^x - 1, 0) \tag{117}$$

7.4.2 An ansatz

Now make the following ansatz

$$v(x,\tau) = e^{\alpha x + \beta\tau} u(x,\tau) \tag{118}$$

with

$$\alpha = -\frac{1}{2}(\gamma - 1), \quad \beta = -\frac{1}{4}(\gamma + 1)^2. \tag{119}$$

This leads to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}. \tag{120}$$

which is the diffusion eq. (or heat eq.). The boundary condition reads

$$u(x,0) =: u_0(x) \overset{!}{=} max(e^{\frac{1}{2}(\gamma+1)x} - e^{\frac{1}{2}(\gamma-1)x}, 0) \tag{121}$$

7.4.3 Solution

The solution is known:

$$\begin{aligned} u(x,\tau) &= \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(s)e^{-(x-s)^2/(4\tau)}ds \\ &= exp\left(\frac{1}{2}(\gamma+1)x+\frac{1}{4}(\gamma+1)^2\tau\right)N(d_1) \\ &\quad -exp\left(\frac{1}{2}(\gamma-1)x+\frac{1}{4}(\gamma-1)^2\tau\right)N(d_2) \end{aligned} \tag{122}$$

with

$$N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}s^2}ds=\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right) \tag{123}$$

and

$$d_1 = \frac{x}{\sqrt{2\tau}}+\frac{1}{2}(\gamma+1)\sqrt{2\tau} \tag{124}$$

$$d_2 = \frac{x}{\sqrt{2\tau}}+\frac{1}{2}(\gamma-1)\sqrt{2\tau}. \tag{125}$$

Inserting into the previous expressions gives the **Black- Scholes formula**:

$$G(S(t),t)=S(t)N(d_1)-Ee^{-r(T-t)}N(d_2), \tag{126}$$

which gives the fair price of the European Call option at time t as a function of the current stock price, the exercise price E and the remaining time $T-t$ until expiry. d_1 and d_2 are given by

$$d_1 = \frac{log(S(t)/E)+\left(r+\frac{1}{2}\sigma_0^2\right)(T-t)}{\sigma_0\sqrt{T-t}} \tag{127}$$

$$d_2 = \frac{log(S(t)/E)+\left(r-\frac{1}{2}\sigma_0^2\right)(T-t)}{\sigma_0\sqrt{T-t}} \tag{128}$$

Interest rate r and volatility σ_0 of the stock are assumed to be known quantities.

7.5 Results

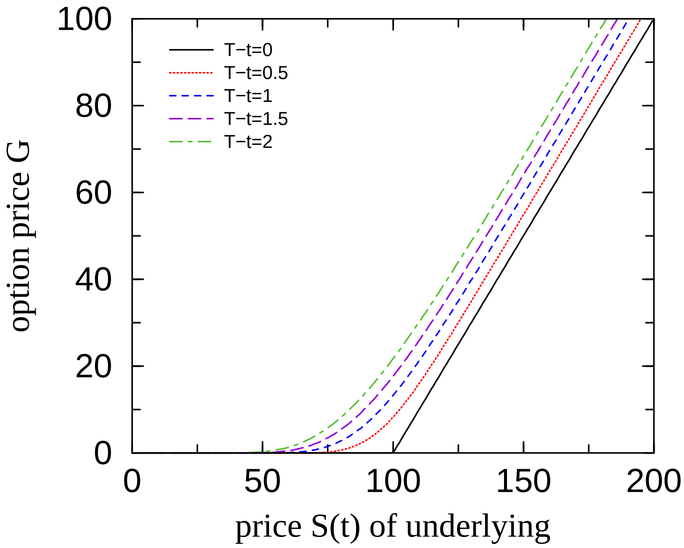


Figure 4: Option price G versus price $S(t)$ at the underlying, $sigma_0 = 0.2$ (20%) and $r = 0.1$ (interest rate 10% per year). Strike price is 100. Different curves are for different time to expiry is $T - t = 0, 0.5, 1, 1.5, 2$ years from bottom to top.

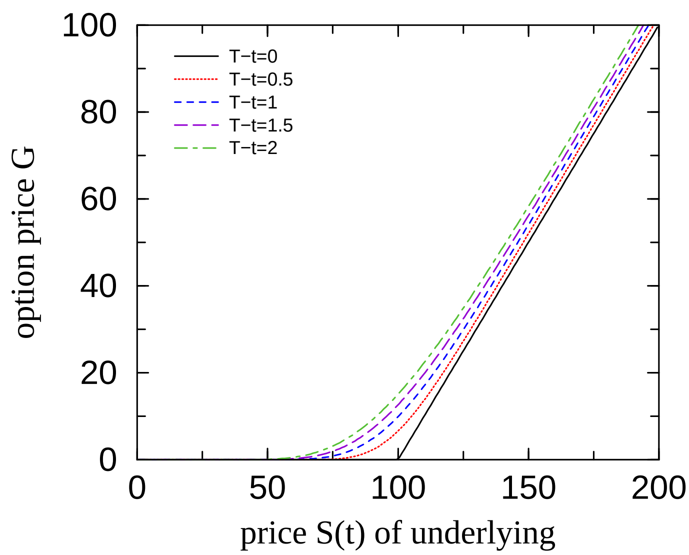


Figure 5: Effect of interest rate: Option price G versus price $S(t)$ at the underlying, $\sigma_0 = 0.2$ (20%) and $r = 0.04$ (interest rate 4% per year). Strike price is 100. Different curves are for different time to expiry is $T - t = 0, 0.5, 1, 1.5, 2$ years from bottom to top.

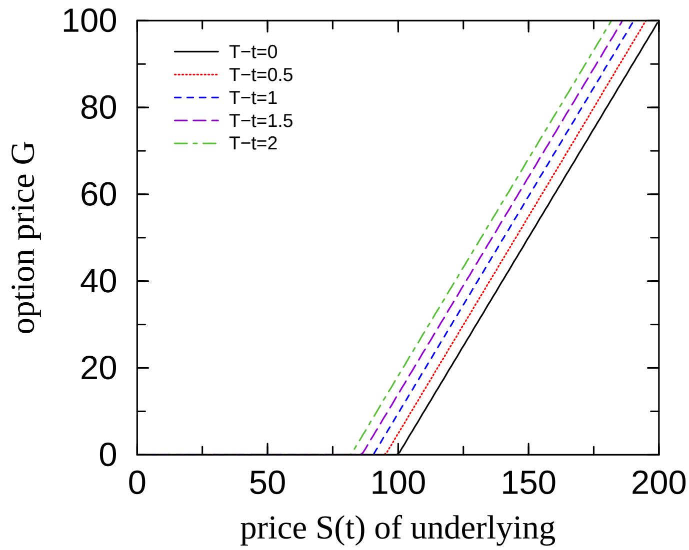


Figure 6: Effect of volatility: Option price G versus price $S(t)$ at the underlying, $\sigma_0 = 0.001$ (0.1%) and $r = 0.1$ (interest rate 10% per year). Strike price is 100. Different curves are for different time to expiry is $T - t = 0, 0.5, 1, 1.5, 2$ years from bottom to top.

7.6 Direct risk-neutral pricing

We note that the mean drift μ_0 does not enter into the BS equation. Thus the fair price of the option does not depend on the mean drift. All investors come to the same estimate of the fair price, independently of their expectation on the mean drift, i.e. bears and bulls will agree on the same fair price.
We can therefore assume a risk-neutral world, and set $\mu_0 = r$ (normally investors would buy shares only of $mu_0 > r$, with r the interest rate of the risk free bond).
The expected value of the option at the expiry date is then given by

$$\hat{E}(max(S(T) - E, 0) \tag{129}$$

with $\hat{E}$ the expectation value in the risk-neutral world. The average is over $S(T)$ (which is random).
S follows a stochastic process

$$dS = \mu_0 S\, dt + \sigma_0 S\, dW, \tag{130}$$

so that $S(T)$ follows a log-normal distribution. Using Ito's lemma one finds that

$$log(S(T) - log(S(0)) \tag{131}$$

follows a Gaussian distribution with

- mean $\left(\mu_0 - \frac{1}{2}\sigma_0^2\right)(T - t)$
- and variance $\sigma_0(T - t)$

Setting $\mu_0 = r$, we can directly compute the expectation value in (129) by integrating over $S(T)$.
The fair price at time t is then obtained by discouting the value $\hat{E}(max(S(T) - E, 0)$ according to the interest rate r:

$$G(S(t), t) = e^{-r(T-t)}\hat{E}(max(S(T) - E, 0)). \tag{132}$$

Computing the integral over $S(T)$ in $\hat{E}(max(S(T) - E, 0))$ then leads to the above Black-Scholes expression (126).

7.7 Determination of parameters

Problem: the BS formula requires the knowledge of the volatility σ_0 of the underlying. Note that the mean drift μ_0 does not enter. So one needs to have an estimate of σ_0. Two ways of doing this:

7.7.1 Historical volatility

Estimate the volatility σ_0 from the past time series of the stock price (say last 90-180 trading days). Then assume that this is a good proxy for the future volatility.

7.7.2 Implied volatility ("Let the market tell you !")

The idea of an 'implied volatility' is based on computing the volatility from the option prices which are actually realised on the market. A realised option price $G(t)$ can be observed on the market, exercise prices and remaining time until expiry $T - t$ are known as well as the interest rate r. Inserting this into the Black-Scholes equation allows one to solve numerically for σ_0.

Often the implied volatility of the underlying is determined from a whole range of different options derived from this underlying asset (with differen running times, initial times, exercise prices ...). In a fully Gaussian world, in which everybody trades according to the Black-Scholes formula, this should always lead to the same volatility estimate σ_0.

In reality this is of course not the case, one observes the so-called 'smile effect'. At fixed price $S(t)$ different options with varying strike prices imply different volatilities. An option is called

- *deep in the money* for $S(t) \gg E$
- *in the money* for $S(t) > E$
- *at the money* for $S(t) \approx E$
- *out of the money* for $S(t) < E$

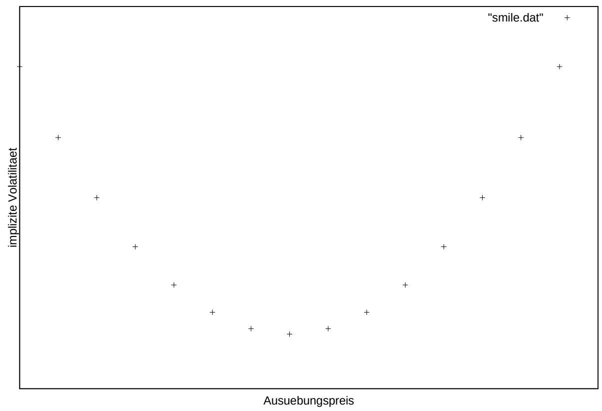


Figure 7: Smile effect: implied volatility versus strike price

- *deep out of the money* for $S(t) \ll E$

References

[1] C.W Gardiner, Handbook of Stochastic Methods, Springer 1985

[2] N.G. van Kampen, Stochastic Processes in Physics and Chemistry, Elsevier Science

[3] P. Wilmott, S. Howison, J. Dewynne, The Mathematics of Financial Derivatives, Cambridge University press (highly recommended book)

[4] J. Hull, Option, Futures and Other Derivative Securaties, Prentice-Hall

[5] M.S. Joshi, The concepts and practice of mathematical finance, Cambridge University Press

[6] F. Black, M. Scholes, The Pricing of Options and Corporate Liabilities, J. Pol. Econ. 81 (1973), 637-654

[7] J.P. Bouchaud, M. Potters, *Theory of financial risk, from statistical physics to risk management*, Cambridge University Press, 2nd edition (!)