PHYS20352: Thermal and Statistical Physics

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Example Sheet 10

1. The one-body partition function of a spin-1 paramagnet in an applied field is given by Q2(a) of Example Sheet 9,

$$Z_1 = 1 + 2\cosh(\beta\epsilon),$$

where $\beta = 1/k_B T$ and ϵ is the energy parameter.
(a) Hence, show that the internal energy per particle of the paramagnet is given by

$$\frac{E}{N} = -\frac{2\epsilon\sinh(\beta\epsilon)}{1 + 2\cosh(\beta\epsilon)},$$

and its heat capacity per particle at a constant applied field is

$$\frac{C}{N} = 2k_B\left(\frac{\epsilon}{k_B T}\right)^2 \frac{2 + \cosh(\epsilon/k_B T)}{[1 + 2\cosh(\epsilon/k_B T)]^2}.$$

(b) Sketch both $E/\epsilon N$ and $C/k_B N$ as functions of $(\epsilon/k_B T)$. Find their exact limits at both low ($T \to 0$) and high ($T \to \infty$) temperatures, and give physical explanations of your findings.

2. (a) The one-body partition function of a system of many independent quantum oscillators at temperature T was calculated in class as

$$Z_1 = \frac{1}{2\sinh(\beta\hbar\omega/2)},$$

where $\beta = 1/k_B T$ and ω is the angular frequency of an oscillator. Derive expressions for the internal energy E and Helmhotz free energy F of a crystal which is modeled as a system of $3N$ independent quantum oscillators with the same angular frequency ω. Show that as $T \to 0$ the internal energy of the crystal is just the zero-point energy of all the oscillators.
(b) The thermal expansion of the crystal can be explained if ω varies with volume as $V^{-\gamma}$, where γ is a constant. Show that in this case the pressure exerted by the lattice vibrations is

$$P = \frac{\gamma E}{V}.$$

(c) Calculate the specific heat (heat capacity per particle) of the crystal at constant volume. Discuss its behaviors at high and low temperature. Compare your results with the classical results of the constant k_B per oscillato.