Edge on-site potential effects in a honeycomb topological magnon insulator

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While the deviation of the edge on-site potential from the bulk values in a magnonic topological honeycomb lattice leads to the formation of edge states in a bearded boundary, this is not the case for a zig-zag termination where no edge state is found. In a semi-infinite lattice, the intrinsic on-site interactions along the boundary sites generate an effective defect and this gives rise to Tamm-like edge states. If a non-trivial gap is induced, both Tamm-like and topologically protected edge states appear in the band structure. The effective defect can be strengthened by an external on-site potential and the dispersion relation, velocity and magnon-density of the edge states all become tunable.

I. INTRODUCTION

Many important phenomena in condensed matter physics are related to the formation of edge or surface states along the boundary of finite-sized materials. Their existence has been commonly explained as the manifestation of Tamm¹ or Shockley² mechanisms. In recent years it has been revealed that the edge states in the so-called topological insulators³ are related to the bulk properties^{4,5}. One such property is characterized by an insulating bulk gap and conducting gapless topologically protected edge states that are robust against internal and external perturbations^{6,7}.

Edge states in topological magnon insulators have also attracted a lot of attention recently^{8–11}. The magnons are the quantized version of spin-waves^{12,13}, which are collective propagation of precessional motion of the magnetic moments in magnets. Since there is no particle movement, magnons can propagate over a long distance without dissipation by Joule heating^{14,15}. Similar to spintronics¹⁶, the study of the edge magnons will enrich the potential of magnonics, exploiting spin-waves for information processing^{17–20}. For this purpose the complete understanding of the edge magnon behavior in different lattice structures and the precise control of their properties are urgently called for.

The magnon hall effect was observed in the ferromagnetic insulator $Lu_2V_2O_7^{21}$, in the Kágome ferromagnetic lattice²², in $Y_3Fe_5O_3$ (YIG) ferromagnetic crystals^{23,24}, and have also been studied in the Lieb^{25} and the honevcomb ferromagnetic lattices²⁶. Interestingly, it has been shown that a ferromagnetic Heisenberg model with a Dzialozinskii-Moriya interaction (DMI) on the honeycomb lattice realizes magnon edge states similar to the Haldane model for spinless fermions²⁶ and the Kane-Mele model for $electrons^{27}$. By a topological approach, it has been shown that a non-zero DMI makes the band structure topologically non-trivial and by the winding number of the bulk Hamiltonian, gapless edge states which cross the gap connecting the regions near the Dirac points has been predicted²⁶. The thermal Hall effect²⁸ and spin Nernst effect²⁷ have also been predicted for this magnetic system. By a direct tight binding formulation in

an strip geometry, it was shown that the edge states in a lattice with a zigzag termination closely resembles their fermionic counterpart only if an external on-site potential is introduced at the outermost sites²⁹. Furthermore, the lattice with armchair termination has additional edge states to those predicted by a topological approach. Such edge states were found to be strongly dependent to edge on-site potentials³⁰. On the other hand, in a semiinfinite ferromagnetic square lattice, a renormalization of the on-site contribution along the boundary gives rise to spin-wave surface states 31-33 and most recent experiments in photonic lattices have observed unconventional edge states in a honeycomb lattice with bearded³⁴, zigzag and armchair³⁵ boundaries, which are not present in the fermionic graphene. In addition, Tamm-like edge states were also observed in a Kágome acoustic lattice³⁶. These unconventional edge states are found to be related to the bosonic nature of the quasi-particles in the lattice whose model hamiltonians contains on-site interaction terms.

In this work, we explore in more detail the magnon edge states in a honeycomb lattice with a DMI and an external on-site potential along the outermost sites. Extending our previous work^{29,30}, we develop a general approach applied to a zig-zag and bearded terminations and we derive analytical expressions for both energy spectrum and wavefunctions. In a lattice with a boundary, the interaction terms along the outermost sites differ from the bulk values. Such a difference plays the role of an effective defect and gives rise to Tamm-like edge states³⁴ the type of edge states generated by an strong perturbation due to an asymmetric termination of a periodical potential¹. In similar fashion, the on-site potential along the boundary plays an important role in the appearance of edge states in bosonic lattices. We found that the effective defect can be strengthened by an external on-site potential and this can be used to tune up the dispersion and velocity of the edge states present in the system. For both boundaries under consideration, we present a simple diagram with which the number of magnon edge states can be predicted. In addition, if a non-trivial gap is induced, the edge state band structure is found to be strongly dependent to the on-site interactions. The tight binding formulation which we have implemented in this work facilitates extraction of analytical solutions of both

energy spectrum and wavefunctions for better physical understanding. All our analytical results are in agreement with direct numerical calculations.

TIGHT-BINDING MODEL ON THE II. HONEYCOMB LATTICE

In this section, we briefly present the general approach for the study of the edge states with an arbitrary external on-site potential and with a DMI.

A. Harper's equation

The bosonic tight-binding Hamiltonian on the honeycomb lattice, derived from a linear spin-wave approximation to the Heisenberg model, is given by

$$H = -JS \sum_{\langle i,j \rangle} \left(a_i b_j^{\dagger} + a_i^{\dagger} b_j - a_i^{\dagger} a_i - b_j^{\dagger} b_j \right) + H_D, \ (1)$$

where a_i and b_j are bosonic operators of the two sublattices, $\langle i, j \rangle$ indicates a nearest-neighbor (NN) coupling with isotropic ferromagnetic coupling constant J (> 0), Sis the spin quantum number from the original Heisenberg model²⁹ and $H_D = H_{D,A} + H_{D,B}$ is the DMI contribution. In particular, $H_{D,A}$ is given by

$$H_{D,A} = iDS \sum_{\langle \langle i,j \rangle \rangle} \varrho_{i,j} \left(a_i a_j^{\dagger} - a_i^{\dagger} a_j \right), \qquad (2)$$

where D is the DMI strength, $\langle \langle i, j \rangle \rangle$ runs over the nextnearest-neighbor (NNN) sites, the hopping term $\rho_{ij} = \pm 1$ depending of the orientation of the two NNN sites³⁷ and $H_{D,B}$ is similar for the *B*-sublattice. The Hamiltonian in Eq. (1) is the bosonic equivalent to the Haldane model³⁸, where the NNN complex hopping in Eq. (2) breaks the lattice inversion symmetry and makes the band structure topologically non-trivial. To analyze the edge states we consider a lattice with an open boundary along the x direction and semi-infinite in the y direction as shown in Fig. (1). In the linear spin-wave approximation, by denoting wavefunctions on two sub-lattices of the honeycomb lattice as $\psi_{A,n}$ and $\psi_{B,n}$, respectively, the Harper's equation provided by the Hamiltonian in Eq. (1) can be written as

$$3\psi_{A,n} - J_1\psi_{B,n} - J_2\psi_{B,n-1} + f_{A,n} = \varepsilon\psi_{A,n}, -J_1\psi_{A,n} - J_2\psi_{A,n+1} + 3\psi_{B,n} - f_{B,n} = \varepsilon\psi_{B,n},$$
(3)

where n is a row index in the y direction perpendicular to the boundary. In the above equation, the DMI is given by $f_{l,n} = J_3 \psi_{l,n} - J_4 (\psi_{l,n+1} + \psi_{l,n-1})$, with l (= A, B) a sublattice index. Furthermore, if k is the momentum in the x direction, the hopping amplitudes for the lattice with a zig-zag boundary are given by: $J_1 = 2\cos(\sqrt{3k/2}), J_2 = 1, J_3 = 2D'\sin(\sqrt{3k}),$ $J_4 = 2D' \sin(\sqrt{3}k/2)$ and D' = D/J. In addition, the

 $\mathbf{2}$



FIG. 1. (Color online) Schematics of the a) zig-zag and b) bearded boundaries on the honeycomb lattice. The sublattices are labeled by A and B. The external on-site potential δ_1 is applied at the outermost sites. Here, n is a index row along the y direction perpendicular to the boundary.

simple replacements of $J_1 \rightarrow J_2$ and $J_2 \rightarrow J_1$ in Eq. (3) provide the corresponding Harper's equation for the lattice with a bearded boundary.

Effective Hamiltonian for the edge states В.

The Harper's equation of Eq. (3) can be simplified if we assume a decaying Bloch wavefunction in the y direction of the form, $\psi_{l,n} = z^n \psi_l$, where l labels each sublattice and the Bloch phase factor z is a complex number³⁹. The eigenequation for the effective Hamiltonian of the edge state can be written with the decaying wavefunction as $H_{ef}\psi_{l,n} = \varepsilon\psi_{l,n}$, where

$$H_{ef} = \begin{bmatrix} 3 + J_3 - J_4\Delta & -w \left(J_1 + J_2 z^{-1}\right) \\ -w^{-1} \left(J_1 + J_2 z\right) & 3 - J_3 + J_4\Delta \end{bmatrix}, \quad (4)$$

and $\Delta = z + z^{-1}$. In the above equation, the factor w takes into account the bearded (w = z) and zig-zag (w = 1) boundaries. The non-trivial solution for the eigenstates of H_{ef} gives rise to the secular equation

$$J_4^2 \Delta^2 - (2J_3 J_4 - J_1 J_2) \Delta - \varepsilon_r^2 + J_1^2 + J_2^2 + J_3^2 = 0,$$
 (5)

where $\varepsilon_r = \varepsilon - 3$. Note that such polynomial in Δ is the same for the both considered boundaries. For a given momentum k and energy ε , the solutions of Eq. (5) are the Bloch phase factors z_{ν} with $\nu = 1, ..., 4$. In particular, for the infinite system, the Fourier transform in the ydirection is the solution $z = e^{\pm \frac{3}{2}ik_y}$ which corresponds to Bloch extended states. In the case of a lattice with a boundary, the solutions of Eq. (5) satisfying $|z_{\nu}| = 1$ determine the bulk band structure [See. Fig. (2)]. The states with $|z_{\nu}| \neq 1$ decay or grow exponentially in space, and they can be used to describe the edge states with the appropriate boundary conditions.

The factors z_{ν} and z_{ν}^{-1} in Eq. (5) always appear in pairs. Since we require a decaying (evanescent) wave from the boundary, setting the condition $|z_{\nu}| < 1$ implies that the general solution for the edge states can be written as a linear combination of the form,

$$\psi_{l,n} = c_1 z_1^n \psi_l^{(1)} + c_2 z_2^n \psi_l^{(2)}, \tag{6}$$

where the coefficients c_i , with i = 1, 2, are determined by the boundary conditions. In the above equation, $\psi_l^{(\nu)}$ with $\nu = 1, 2$ is an eigenvector of H_{ef} corresponding to the ν -th decaying solution of Eq. (5). To obtain the edge state energy spectrum, the wavefunctions given by Eq. (6) must satisfy the boundary conditions. This will be described in the following sections.

III. BOUNDARY CONDITIONS AND THE EDGE STATES

In this section, the boundary conditions for both zigzag and bearded boundaries are obtained. By the secular Eq. (4) and the boundary conditions, we derive the analytical expressions for the edge state energy spectrum and wavefunctions for non-zero DMI. The analytical solutions with zero DMI are obtained in the appendix A.

A. Zig-zag boundary

In our previous work²⁹, we have derived the equations for the energy and the wavefunctions considering a fixed on-site potential $\delta_1 = 1$, where the edge state energy spectrum and the wavefunctions closely resembles the fermionic graphene. Here, we first summarize, then extend the formalism to arbitrary values of the external on-site potential.

Due to the open zig-zag boundary, the on-site potential along the boundary is different from that in the bulk. The Harper's equation of Eq. (3) at n = 1 must be modified. Considering the missing bonds along the outermost Asite, the coupled Harper's equation at n = 1 is written as,

$$(2 - \delta_1) \psi_{A,1} - J_1 \psi_{B,1} + f_{A,1} = \varepsilon \psi_{A,1},$$

$$3\psi_{B,1} - (J_1 \psi_{A,1} + J_2 \psi_{A,2}) - f_{B,1} = \varepsilon \psi_{B,1}, \quad (7)$$

where the external on-site potential δ_1 is introduced and $f_{l,1} = J_3 \psi_{l,1} - J_4 \psi_{l,2}$. In the the above equation, the total energy at each sublattice is given by the on-site contribution (first term), the NN contribution (second term) and the DMI (third term). From Eqs. (3) and (7), we obtain the zig-zag boundary conditions

$$(1 - \delta_1) \psi_{A,1} - J_4 \psi_{A,0} = 0, \tag{8}$$
$$\psi_{B,0} = 0,$$

for the edge state wavefunctions in Eq. (6). Unlike the equivalent fermionic model⁴⁰ where the wavefunctions of both sub-lattices vanish at n = 0, Eq. (8) contains the on-site contribution at n = 1. As we will shown in the following sections, such contribution have important effects in the band structure of the edge states. From Eqs. (6) and (8) the non-trivial solution for the coefficients c_i provides the following self-consistent equation for the edge state energy spectrum,

$$\varepsilon = 3 + J_3 - J_4 \left\{ \frac{\left[(\delta_1 - 1) J_1 - J_2 J_4 \right] (z_1 + z_2) + \left[J_2 \left(\delta_1 - 1 \right) + J_1 J_4 \right] (1 - z_1 z_2)}{(\delta_1 - 1) J_1 z_1 z_2 - J_2 J_4} \right\}.$$
(9)

In the above equation, z_1 and z_2 are two decaying solutions of Eq. (5). The corresponding edge state wavefunctions are given by

$$\psi_{A,n} = c_1 \left(z_1^n - \alpha z_2^n \right) \psi_A^{(1)}, \psi_{B,n} = c_1 \left(z_1^n - z_2^n \right) \psi_B^{(1)},$$
(10)

where c_1 is a normalization term, and

$$\alpha = \frac{(1 - \delta_1) z_1 - J_4}{(1 - \delta_1) z_2 - J_4},\tag{11}$$

contains the contribution of the external on-site potential in the wavefunction. For a given momentum k, external potential δ_1 and non-zero DMI, Eq. (9) is an implicit equation for the energy ε and can be solved numerically. Equations (5), (9) and (10) provide a full description for the edge state energy spectra and their corresponding wavefunctions, which will be described in the Sec. IV.

B. Bearded boundary

Similar to the zig-zag case, by modifying the Harper's equation at n = 1 to take into account the missing sites, the boundary conditions for the wavefunctions in Eq. (6) are given by

$$(2 - \delta_1) \psi_{B,1} + J_4 \psi_{B,0} = 0, \qquad (12)$$

$$\psi_{A,0} = 0.$$

From Eqs. (6) and (12) the non-trivial solution for the coefficients c_i can also be obtained. We find that the simple replacements: $J_1 \rightarrow J_2$, $J_2 \rightarrow J_1$, $J_3 \rightarrow -J_3$, $J_4 \rightarrow -J_4$ and $\delta_1 \rightarrow \delta_1 + 1$ in Eq. (9), provide the self-consistent equation for the edge state energy spectrum. The wavefunctions satisfying the boundary conditions of Eq. (12) are given by

$$\psi_{A,n} = c_1 \left(z_1^n - z_2^n \right) \psi_A^{(1)}, \psi_{B,n} = c_1 \left(z_1^n - \alpha' z_2^n \right) \psi_B^{(1)},$$
(13)



FIG. 2. (Color online) a) Edge state dispersion relations induced by δ_1 at the zig-zag open boundary are shown for $\delta_1 = 0.2, 0.6, 0.8$ and 1.0 (from the curved to the flat band). For $\delta_1 = 0$ there are not edge states. The blue region is the bulk continuum where all the factors $|z_v| = 1$ in Eq. (5). b) Modulus of the decaying factors for the corresponding edge states, here ± 1 is the sign of z_1 .

where c_1 is a normalization term and

$$\alpha' = \frac{(2-\delta_1)z_1 + J_4}{(2-\delta_1)z_2 + J_4}.$$
(14)

Equations (5) and (12) together with Eq. (13) provide a full description for the edge state energy spectrum and wavefunctions. For an arbitrary external on-site potential and zero DMI, the k-dependence of $\varepsilon(k)$ and the explicit solutions for the decaying z factors are obtained in the appendix A.

IV. ENERGY SPECTRUM AND WAVEFUNCTIONS

A. Zero DMI

In a fermionic honeycomb lattice with a boundary, it is well known that there are flat edge states connecting the two Dirac points, $\mathbf{K} \to \mathbf{K'}$, in a lattice with a zig-zag boundary⁴¹, whereas in a lattice with a bearded boundary⁴², the flat edge state is connecting the complementary region, $\mathbf{K'} \to \mathbf{K}$. In the equivalent bosonic models, we expect different situations due to the contribution of the on-site interactions along the boundary sites.

1. Zig-zag boundary

For a zig-zag boundary, in absence of external on-site potentials and zero DMI, the solutions of Eq. (5) and (7) provide bulk states with $z^2 = 1$ and energy $\varepsilon = 2 \pm |J_1|$ and no edge state is found. However, a Tamm-like edge state can be induced if the effective defect is strengthened by turning on the external on-site potential, δ_1 . In Fig. (2) the energy spectra and the decaying factors of the induced edge states are shown for different values of δ_1 . As the external on-site potential is increasing $(\delta_1 \to 1)$,



FIG. 3. (Color online) Spin density profile for k = 1.41 and external on-site potential a) $\delta_1 = 0.2$ and b) $\delta_1 = 0.4$. For clarity, the magnitudes on the edge are held constant. The magnitudes of the spin density are proportional to the radius of each circle with a phase given by $e^{in\theta_l} = \pm 1$.

the branch becomes flatter [Fig. (2a)] and from the edge state wavefunctions,

$$\begin{pmatrix} \psi_{A,n} \\ \psi_{B,n} \end{pmatrix} = z^n \begin{pmatrix} z^{-1} \\ \frac{1-\delta_1}{J_2} \end{pmatrix}, \tag{15}$$

the magnon density is found to be more localized in a single lattice [See Fig. (3)]. In the above equation, the decaying factor z is a real number. For a wide ribbon^{4,5}, the edge state energy spectra is double degenerated and since the magnon velocity is the slope of the energy spectrum, the magnons are moving in the same direction at opposite edges, as illustrated in Fig. (4a). As shown in Fig. (2), as δ_1 is increased the slope (and the edge magnon velocity) is reduced until $\delta_1 = 1$ where the edge state becomes non-dispersive.

If the external on-site potential is increased, the number of edge states changes as well as their shape. Depending on the external on-site potential strength, a zig-zag termination can have two edge states at each boundary. In the decaying factor diagram of the Fig. (5a), each edge state has a corresponding z_1 or z'_1 decaying factor. For $0 < \delta_1 < 2$, there is a single decaying factor between the Dirac points [see also Fig. (2)] and from Eq. (15) it is straightforward to show that the edge state in this region is mainly localized at the A sublattice. As is shown in the Fig. (5a), for $\delta_1 > 2$ there are two edge states, the first one, corresponding to z'_1 , is defined over all the Brillouin zone with energy spectra over the bulk bands (due to the strong external on-site potential). The second edge state, corresponding to z_1 , is defined in the region K > k > K' as in the bearded graphene. Such edge state has a magnon density mainly localized at the B sublattice with energy spectrum between the bulk bands. If the external on-site potential is even stronger, $\delta_1 \gg 2$, the system effectively shows the band structure of a bearded termination plus a high energy Tamm-like edge state. Moreover, as we mentioned before, in absence of external on-site potential ($\delta_1 = 0$) there are not edge states. At $\delta_1 = 2$, there are not edge states either. This can be observed in the diagram of the Fig. (5a), where at such value, $|z_1| = |z_1'| = 1$ for all values of k. At the transition lines (dashed) the modulus of the decaying factors



FIG. 4. (Color online) Schematic illustrations of the edge states propagation with the same momentum k. The arrows represent the edge magnon velocity. At each boundary we can have, a) a single edge magnon, b) two edge magnons with opposite and different velocities, c) two edge magnons moving in the same direction at different velocity and d) chiral edge states.

reaches the unity and the edge states are indistinguishable from the bulk bands.

The magnon excitations in a ferromagnetic lattice can be viewed as a synchronic precession of the spin vectors. The sign of the wavefunctions in Eq. (15) can be related to the spin precession in successive n rows and the wavefunction modulus to the radius of precession which decrease as n increases. If we write the phase of the wavefunction as $e^{in\theta_l} = sgn(\psi_{l,n})$, then, for a given k and $0 < \delta_1 < 2$, the synchronic precession of the spins in successive n rows is in anti-phase ($\theta_l = \pi$, optic-like) if $k < k_0 (= \pi/\sqrt{3})$ and in-phase ($\theta_l = 0$, acoustic-like) if $k > k_0$. Furthermore, at the same n, the spins at different sub-lattices are precessing in anti-phase for $k < k_0$ and in-phase for $k > k_0$ [See Fig. (3a)]. At the transition point k_0 , the edge state energy is $\varepsilon_0 = 2 + \delta_1$ and the decaying factor is zero as shown in Fig. (2b). Hence, for $\delta_1 \neq 0, 2$, and by Eq. (15), the magnon is completely localized at the edge site.

2. Bearded boundary

We now consider a bearded termination. As shown in Fig. (1b), the outermost site has two missing bonds and the effective defect is stronger than the corresponding to a zig-zag boundary. Contrary to the fermionic equivalent, the on-site terms provided by Eq. (1) change substantially the edge state band structure. This is shown in Fig. (6a), where for $\delta_1 = 0$ there are two edge state energy bands as given by Eq. (A6), the first one between the Dirac points (dot-dashed, black line) and the second one below the lower bulk bands (dashed, black line). Such edge states are defined in a region in k completely different to their fermionic equivalent 42,43 . As shown in Fig. (6b), the edge state below the bulk bands is defined over all the Brillouin zone, except at $k = 0, 2\pi\sqrt{3}$, where the decaying factor reaches the unity and the edge state is indistinguishable from the bulk bands. As is shown in the Appendix A, the edge state wavefunctions are given by,

$$\begin{pmatrix} \psi_{A,n} \\ \psi_{B,n} \end{pmatrix} = z^n \begin{pmatrix} \frac{2-\delta_1}{J_1} \\ z^{-1} \end{pmatrix}, \tag{16}$$



FIG. 5. (Color online) Decaying factor diagram for the edge states in a honeycomb lattice with a) zig-zag and b) bearded termination for zero DMI. The number of decaying factors with |z| < 1 is the number of edge states in the corresponding region. The (dashed) lines dividing each region are the points where both decaying factors reach the unity.

where z is a real number. In the above equation $z = z'_1$ for the edge state below the lower bulk bands and $z = z_1$ for the edge state between the Dirac points [see Fig. (5)]. In Figs. (6c) and (6d), we plot the magnon density, $|\psi_{l,n}|^2$, for both edge states at different momentum. Note that the edge states are localized in different sub-lattices.

Discussion of some interesting features about these edge states are in order here. From Fig. (6a), for $\delta_1 = 0$ the slope the edge state energy spectra is positive if $k < k_0$ and negative if $k > k_0$. For a wide ribbon, each edge band is doubly degenerated, hence, the magnons are moving in the same direction (with different energy) at each boundary, as illustrate in Fig. (4c). The fact that both edge states are strongly localized in different sublattices can be explained if we consider the edge by itself a defect. By a closer inspection of the wavefunctions in Eq. (16), the edge state below the lower bulk bands is mainly localized along the boundary B sites due to the strong attractive potential generated by the missing bonds. The edge state between the bulk bands is mainly localized along the A sublattice due to the presence of the outermost B site. In consequence, the outermost Bsite plays a double role acting as an effective defect to host an edge state and contributing to the formation of the edge state between the Dirac points.

The number of edge states is determined by the number of solutions of Eq. (A3) with modulus lower than one and the edge state dispersion can be tuned in all the Brillouin zone with small changes of the external on-site potential. This is shown in the decaying factor diagram in Fig. (5b), where the dashed lines separate the regions in which each edge state is defined. In the region, $0 \le \delta_1 < 1$, there are always two edge states (for z'_1 and z_1). If $\delta_1 = 0$, the first edge state is defined over all the Brillouin zone with $|z'_1| < 1$, and the second one between the Dirac points with $|z_1| < 1$. As δ_1 is increased both edge states gradually merge with the bulk bands. For $\delta_1 = 2$, there is a single edge state with a momentum in the region, $\mathbf{K} > \mathbf{k} > \mathbf{K'}$. This edge state is the flat band in Fig. (6a), (dotted, green line) where the energy



FIG. 6. (Color online) a) Bulk (blue region) and edge state energy spectra for $\delta_1 = 0$, (black, dashed and dot-dashed lines), $\delta_1 = 1.8$ (red, continuous lines) and $\delta_1 = 2$ (green, dotted lines). In b) we show their corresponding decaying factors. The magnon density profile is shown for the edge states with $\delta_1 = 0$ at c) k = 1.40 and d) k = 0.96. Here, the magnitudes of the spin density are proportional to the radius of each circle with a phase given by $e^{in\theta_l} = \pm 1$.

spectra closely resembles the fermionic graphene. If the external on-site potential is increased further, $\delta_1 \gg 2$, the hopping between sites at n = 1 is almost suppressed and the system effectively shows the band structure of a zig-zag termination plus and a high energy Tamm-like edge state along the boundary sites.

Another important characteristic provided by the explicit form of the wavefunction in Eq. (16) is given by the phase of the spin precession in successive rows. As discussed in the previous section, the sign of the decaying factor determines whether the phase of the edge state is optic-like or acoustic-like. As is described in the appendix A, there are two decaying factors and their sign reveals that the behavior of the phase in successive rows is different in both edge states. In particular for $\delta_1 = 0$, the decaying factor of the edge state connecting the Dirac points is negative if $k < k_0$, the spin precession in successive lattice sites is hence in anti-phase (optic-like). However, the decaying factor of the edge state below the lower bulk bands is positive if $k < k_0$, and the spins in two successive rows are in-phase (acoustic-like). This provides us two ways to distinguish these edge states, either by their energy or by their phase difference in successive rows.

Experimentally, the first observation of edge states in a honeycomb lattice with bearded boundaries has been achieved in optical lattices³⁴. Apart from the typical band structure, additional edge states have been observed near the Van Hoove singularities. As is shown in Fig. (6a) for our model, similar edge states are obtained for an external on-site potential of $\delta_1 = 1.8$. Here a nearly flat band plus two highly dispersive edge states near the Van Hoove singularities (continuous, red lines) are obtained. As pointed out in the reference³⁴, the origin of such edge states is also related to the effective defect generated by the on-site potential along the boundary sites.

B. Non-zero DMI

A non-zero DMI breaks the lattice inversion symmetry and a non-trivial gap is induced in the spin-wave excitation spectra. By a topological approach with the wavefunctions for the infinite system, the Chern number predicts a pair of counter propagating modes²⁶ along the boundary of the finite system. However, the topological approach does not provide the detailed properties of the edge states and also does not take into account the onsite potential along the boundary sites, which, as we will show in this section, has important effects in the band structure of the edge states.

1. Zig-zag boundary

We first consider a zig-zag boundary. The energy bands are obtained by the solutions of the self-consistent Eq. (9) with the decaying factors provided by Eq. (5). In Fig. (7a) we show the energy bands for a DMI strength of D = 0.1J. The blue regions correspond to the bulk spectra where all the factors $|z_{\nu}| = 1$. The bands which transverse the gap are the spectra of the edge states for different values of δ_1 . For completeness, we also include the energy spectra for the edge state at the opposite edge (at large n), without external on-site potential. Contrary to the description by a topological approach 26,27 , the edge state is not connecting the regions near the Dirac points. As is shown in Fig. (7a), for $\delta_1 = 0$ (red, continuous line) the intrinsic on-site potential along the boundary pull the edge state within the bulk gap to a lower energy region, just over the lower bulk bands. Furthermore, a new edge state near the Van Hoove singularities is revealed in the band structure. As is shown in the zoomed region of Fig. (7b), around k_0 there are two edge states (at each boundary), over and below the lower bulk band. The edge state over the bulk bands has a topological origin and the edge state below is a Tamm-like edge state.

In general, the edge states depend on two decaying factors, as described in Eq. (10). In Fig. (7c) their typical behavior can be observed: if we move away from k_0 , while one factor decreases to zero the another one approaches to a critical value (merging point) where it reaches the unity. In this situation, one component of the edge state wavefunction becomes an extending wave (bulk wave) and the edge state is indistinguishable from the bulk bands. However, as is shown in Fig. (7d), in the region $k > k_0$, while one decaying factor reaches the unity the second one has enough strength to modified the bulk band structure [arrows in Fig. (7b) and Fig. (7d)], in this situation the edge state has energy within the continuum^{44,45}. For $\delta_1 = 0$ (and $D \neq 0$), the edge band within the bulk gap has a negative slope while the novel edge band below the lower bulk band has a positive slope. Therefore, the magnons are moving in opposite directions at the same boundary, as shown in Fig. (4b). If the external on-site potential is slightly increased the



FIG. 7. (Color online) a) Energy spectrum of a zig-zag honeycomb lattice for D = 0.1 J. The lines connecting the upper and lower bulk bands are the edge states for different values of δ_1 , the dashed (black) lines are the edge states at the opposite edge. As shown in b) for $\delta_1 = 0$ there are additional edge states below the lower bulk bands [dashed square in a)]. In c) the decaying factors of the corresponding edge states in a) is shown. d) Decaying factors ($\delta_1 = 0$) of the edge state below the lower bulk band in a) and b). The energy spectra in b) reveals an edge state with energy within the bulk bands (black arrow), the magnitude of its corresponding decaying factor is given by the arrow in d)

Tamm-like edge magnon merges with the bulk and the magnon propagation is chiral, as is shown in Fig. (4d).

As is shown in Fig. (7a), as the external on-site potential increases, the slope of the energy spectra decreases. In particular, for $\delta_1 = 1$ (uniform case) the energy spectrum closely resembles the fermionic graphene with merging points near the Dirac points and with the magnons moving in opposite directions at different boundaries, as illustrated in Fig. (4d). In Fig. (7c) the modulus of the decaying factors is shown for different values of the external on-site potential. Here, as δ_1 increases, the merging points approaches by the left to the K and K' points and the asymmetry around k_0 is reduced. In the finite region [Fig. (7c)] around k_0 , we have $|z_1| = |z_2|$ and from Eq. (5) it is evident that the decaying factors are complex conjugates to each other. At certain momentum both decaying factors become real and they are not longer identical and, as we mentioned before, while one factor increases the another one decreases. The region around k_0 , where the edge states are complex conjugates to each other, is defined for a non-zero DMI and is located within the bulk gap. Its boundaries in the k space are given by the discriminant of Eq. (5) and is independent of the boundary conditions. If the spectrum of an edge state crosses this region, their corresponding wavefunction becomes complex.



FIG. 8. (Color online) a) Energy spectrum of a bearded honeycomb lattice. The blue region is the gapped bulk spectra with D = 0.1J. For $\delta_1 = 0$, the continuous (red and purple) lines are the edge states. By completeness, we also include the edge states at the opposite edge, dot-dot-dashed (green) lines. For $\delta_1 = 2$ there is a single edge state (black, dotted line). In b) we plot the modulus of the two decaying factors for the edge state connecting the bulk bands at different values of δ_1 . In c) the decaying factors for $\delta_1 = 0$ are shown for the edge state below the lower bulk bands.

2. Bearded boundary

We now consider a bearded termination with a nonzero DMI and arbitrary external on-site potential. The solutions can be obtained by the self-consistent equation provided by Eq. (12) and the wavefunctions by Eq. (13). As is shown in Fig. (8) for $\delta_1 = 0$, there is an edge state crossing the gap (red, continuous line) and an edge state below the lower bulk bands (purple, continuous line). Note that the non-zero DMI changes the shape of the edge magnon spectrum. In fact, the edge state within the gap has a negative slope except near the K point where is almost flat. The edge state energy spectrum below the lower bulk bands has a maximum point where its slope changes. Before such point and out from the almost flat region, the propagation is like in Fig. (4b) where, for a fixed momentum and at the same boundary, the magnons are moving in different directions. On the other hand, as is shown in Fig. (8a), to the right of the K' point, there are two edge bands with negative slope (red and purple continuous lines) and a single edge band with negative slope at the opposite boundary (green, dotdot-dashed line), hence the magnons are moving in the same direction at both edges.

The effective defect due to the missing bonds is strong in the bearded boundary, where the edge state energy spectra are distinct to their fermionic equivalent. As shown in Fig. (8b) the edge state within the bulk gap (red, continuous line) is defined in a region to the right of the K point. The edge state below the lower bulk bands is defined over the whole Brillouin zone and since its origin is due to the effective defect discussed in the previous section, it is not sensitive to small changes in the DMI strength. In Fig. (8c) the decaying factors for this edge state are shown. The curves are almost symmetric around k_0 and since the decaying factors are real, the wavefunction decays exponentially to the inner bulk sites 29,40 . As discussed in the previous section, as we move away from k_0 , one decaying factor approaches to the unity while the another one decreases. Note that Fig. (8b) is similar to Fig. (7b) except that the plots are tilted to opposite sides. Here, as the external on-site potential increases, the merging points approach to k_0 . In particular for $\delta_1 = 2$, the edge state has an energy spectrum connecting the Dirac points (black, dotted line in Fig. (8a)). However, in contrast with their fermionic equivalent, around k_0 (Van Hoove singularity) there is a small region in which a highly dispersive (and almost indistinguishable) edge state is also defined, (black-dotted line in Fig. (8a) and (8b)). If $\delta_1 \gg 2$, as in the case for D = 0, the system shows the band structure of a zigzag termination plus and a high energy Tamm-like edge state.

V. CONCLUSIONS

We have studied the on-site potential effects in the magnon edge states in a honeycomb ferromagnetic lattice with zig-zag and bearded boundaries, extending our earlier work on fixed value of the on-site potential 29 . For zero DMI, the relation between the formation of the Tamm-like edge states and the effective defect due to the on-site potential along the outermost sites has been demonstrated. For non-zero DMI, we have found that the edge state energy spectra is modified due to the missing bonds along the boundary sites and their distribution in the momentum space is different to that predicted by a topological approach. For both zig-zag and bearded boundaries and for zero and non-zero DMI, the edge state properties have been discussed and Tamm-like edge states have been revealed. We have found that the Tamm-like and the topologically protected edge states are tunable by modifying the external on-site potential and the DMI. Furthermore, the analytical expressions for the edge state energy spectrum and their corresponding wavefunctions obtained have give us complete understanding of the edge state properties. We believe that our results may explain the unconventional edge states recently found in $optical^{34,35}$ and $acoustic^{36}$ lattices and motivate new experiments in bosonic topological insulators.

The interesting properties of the honeycomb lattice may be experimentally accessible through engineered spin structures on metallic surfaces⁴⁶, using ultra-cold bosonic atoms trapped in optical lattices⁴⁷, photonic lattices^{48,49}, etc. Therefore, the distribution of the edge magnons, the spin-density and their dependence with the DMI strength and external on-site potentials as presented in this paper could be useful for experiments in small sized mono-layers, thin film magnets or artificial lattices.

Finally, we like to point out that a recent work on a system of two-interacting bosons (doublon) in the Haldane model on the honeycomb lattice has derived an effective Hamiltonian similar to that of Eq. (1) and numerical solutions of an edge state similar to that of Fig. (8), have been found for a bearded boundary⁵⁰. The dispersive edge states similar to that of Fig. (2) for Dirac magnons in a honeycomb ferromagnet have also been produced in Ref.⁵¹. Both these works have confirmed the results of our general approach with analytical solutions for both the energy spectra and wavefunctions presented here, which is an extension of our earlier work^{29,30}.

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Appendix A: Analytical solutions for D = 0

In this appendix, we derive the edge state energy spectrum and wavefunctions for a semi-infinite ferromagnetic honeycomb lattice with a bearded boundary, in absence of DMI and with an arbitrary external on-site potential δ_1 . From Eq. (5) for D = 0, the characteristic equation of the Hamiltonian in Eq. (4) is given by,

$$(3-\varepsilon)^2 - J_1^2 - J_2^2 - J_1 J_2 \left(z + z^{-1}\right) = 0.$$
 (A1)

For a fixed value of k, the above equation relates the decaying factor z with the energy ε . From the Harper's equation of Eq. (3) with the replacements $J_1 \rightarrow J_2$, $J_2 \rightarrow J_1$ and taking into account the missing bonds, the additional equation for the edge state at n = 1 is written as,

$$(3-\varepsilon)\left(1+\delta_1-\varepsilon\right)-J_1J_2z+J_2^2=0.$$
 (A2)

Here, both Eqs. (A1) and (A2) provide us a complete set of equations for the decaying factor and the energy spectrum. Therefore, for an arbitrary external on-site potential, δ_1 , the decaying factor satisfy,

$$az^2 + bz + c = 0, (A3)$$

where, $a = (-2 + \delta_1)^2 J_2$, $b = J_1 \left[(-2 + \delta_1)^2 - J_1^2 \right]$ and $c = -J_1^2 J_2$. Explicitly we obtain,

$$z_1^{(\prime)} = \frac{-\left(\delta_b^2 - J_1^2\right)^2 J_1 \pm |J_1| \sqrt{\left(\delta_b^2 - J_1^2\right)^2 + 4\delta_b^2}}{2\delta_b^2}, \quad (A4)$$

where $\delta_b = -2 + \delta_1$, $J_2 = 1$ and $z_1(z'_1)$ the solutions corresponding to each sign. On the other hand, the edge state energy spectrum satisfy,

$$a_1\varepsilon_r^2 + b_1\varepsilon_r + c_1 = 0, (A5)$$

where, $\varepsilon_r = (\varepsilon - 3) - (-2 + \delta_1)$, $a_1 = (-2 + \delta_1)J_1$, $b_1 = b$ and $c_1 = -(-2 + \delta_1)J_1J_2^2$. For the edge state energy spectra the two solutions are given by,

$$\varepsilon^{\pm} = \frac{6\delta_b + \delta_b^2 + J_1^2 \pm sgn\left(J_1\right)\sqrt{\left(\delta_b^2 - J_1^2\right)^2 + 4\delta_b^2}}{2\delta_b}.$$
(A6)

From the above equation and by a closer inspection of the decaying factors given by Eq. (A4) two edge states can be defined. The wavefunction satisfying the boundary condition,

$$(2 - \delta_1) \psi_{B,1} - J_1 \psi_{A,0} = 0, \tag{A7}$$

can be written as,

$$\psi_{l,n} = z_1^n \begin{pmatrix} \frac{2-\delta_1}{J_1} \\ z_1^{-1} \end{pmatrix}, \tag{A8}$$

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where the decaying factor z_1 is given by Eq. (A4). At $k_0 = \pi/\sqrt{3}$, the edge states are completely localized at the boundary sites with energy,

$$\varepsilon_{k_0}^{\pm} = \frac{1}{2} \left(6 + \delta_b \right) \pm \sqrt{4 + \delta_b}.$$
 (A9)

In particular, as in graphene, for $\delta_1 = 2$ in Eq. (A3), a single decaying factor, $z_1 = -J_2/J_1$ with a corresponding flat energy band $\varepsilon = 3$ in Eq. (A5), are obtained. Following the same procedure, the analytical form of the decaying factor and the edge state energy spectrum for a zig-zag boundary can also be obtained.

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