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Universal behaviour of four-body systems from a functional renormalisation group

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Results from:

Jaramillo Avila and Birse, arXiv:1506.04949
Jaramillo Avila and Birse, arXiv:1304.5454
Birse, Krippa and Walet, arXiv:1011.5852
Krippa, Walet and Birse, arXiv:0911.4608

Background

Ideas of effective field theory and renormalisation group

- well-developed for few-nucleon and few-atom systems
- rely on separation of scales
- Wilsonian RG used to derive power counting

$\rightarrow$ classify terms as perturbations around fixed point (or limit cycle)

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Many unsuccessful attempts to extend to dense matter

- but no separation of scales
- other EFT's for interacting Fermi systems exist
  (Landau Fermi liquid, Ginsburg-Landau theory)
- but parameters have no simple connection to underlying forces

EFTs based on contact interactions

- not well suited for standard many-body methods

$\rightarrow$ switch to lattice simulation [Lee, Meissner *et al*]
  or look for some more heuristic approach

- based on field theory
- can be matched onto EFT's for few-body systems
  (input from 2- and 3-body systems in vacuum)

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Explore functional renormalisation group ("exact" RG)

- based on Wilsonian RG approach to field theories
- successfully applied to various systems in areas from condensed-matter physics to quantum gravity
[version due to Wetterich (1993)]

Outline

- Functional RG
- Spin-$\frac{1}{2}$ fermions
  - Dimer-dimer scattering
- Bosons
  - Efimov physics
  - 4-body systems

Functional RG

Version based on Legendre-transformed effective action $\Gamma[\phi_c]$
(generating function for 1-particle-irreducible diagrams)

- evolves with scale k according to

$$\partial_k \Gamma = -\frac{i}{2}\,\mathrm{STr}\left[(\partial_k R)\left(\Gamma^{(2)} - R\right)^{-1}\right] \quad \text{where} \quad \Gamma^{(2)} = \frac{\delta^2\Gamma}{\delta\phi_c\delta\phi_c}$$

- $R(q,k)$: regulator function suppresses modes with $q \lesssim k$

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Functional differential equation for Γ

- work with truncated ansatz
- local action expanded in powers of derivatives
  (cf low-energy EFTs, but don't know *a priori* if we have a consistent power counting)

Derivative expansion may be good at starting scale K

- use power counting of EFT to determine relevant terms
  (or use this RG to find that power counting in scaling regime)
- but no guarantee that it remains good for $k \to 0$
  (can't be for scattering amplitudes at energies above threshold: cuts $\to$ nonanalytic behaviour)

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To extend validity of expansion:

- add fields to decribe low-energy excitations:
  (dimers, trimers, phonons, ...)
- make consistency checks:
  stability against adding extra terms to ansatz
  stability against changes in form of regulator

$\to$ use this to optimise choice of regulator [Litim, Pawlowski]

Two species of fermion

Fermion “atom” field: $\psi(x)$ (spin-$\frac{1}{2}$ atoms or neutrons)
Boson “dimer” field: $\phi(x)$ (strongly interacting pairs)
Local (nonrelativistic) ansatz for action in vacuum: 2-body sector

$$
\begin{aligned}
&\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger;k] \\
&= \int \mathrm{d}^4x \Bigg[\psi(x)^\dagger\left(\mathrm{i}\,\partial_0 + \frac{\nabla^2}{2M}\right)\psi(x) \\
&\qquad + Z_\phi(k)\,\phi(x)^\dagger\left(\mathrm{i}\,\partial_0 + \frac{\nabla^2}{4M}\right)\phi(x) - u_1(k)\,\phi(x)^\dagger\phi(x) \\
&\qquad - g\left(\frac{\mathrm{i}}{2}\,\phi(x)^\dagger\psi(x)^{\mathrm{T}}\sigma_2\psi(x) + \mathrm{H\,c}\right)\Bigg]
\end{aligned}
$$

g: AA$\to$D coupling
$u_1(k)$: dimer self-energy (u_1/g^2: only physical parameter)
$Z_\phi(k)$: dimer wave-function renormalisation

Regulators

- fermions: sharp cutoff on 3-momentum

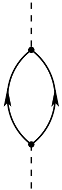
$$R_F(\boldsymbol{q}, k) = \frac{k^2 - q^2}{2M}\,\theta(k - q)$$

- pushes states with $q < k$ up to energy $k^2/2M$
- nonrelativistic version of “optimised” cutoff [Litim (2001)]
- fastest convergence at this level of truncation
- bosons

$$R_B(\boldsymbol{q}, k) = Z_\phi(k)\,\frac{(c_B\,k)^2 - q^2}{4M}\,\theta(c_B\,k - q)$$

- c_B: relative scale of boson cutoff
- optimised choice $c_B = 1$ [cf Pawlowski (2007)]
  (no mismatch between fermion and boson cutoffs)

2-body sector: evolution equations from one "skeleton" diagram



(need to insert $\partial_k \mathbf{R}_F$ on one internal line)

Expand in powers of energy $\rightarrow \partial_k u_1, \partial_k Z_\phi$

3-body sector: AD contact interaction

$$\Gamma[\psi, \psi^\dagger, \phi, \phi^\dagger; k] = \cdots - \lambda(k) \int d^4x\, \psi^\dagger(x)\phi^\dagger(x)\phi(x)\psi(x)$$

4 diagrams in evolution equation

4-body sector: DD→DD, DD→DAA, DAA→DAA terms

[Birse, Krippa and Walet (2010); cf Schmidt and Moroz (2009): bosons]

$$\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger;k] = \cdots - \int d^4x \left[\frac{1}{2}u_2(k)\left(\phi^\dagger\phi\right)^2 + \frac{1}{4}v(k)\left(\mathrm{i}\,\phi^{\dagger 2}\phi\,\psi^{\mathrm{T}}\sigma_2\psi + \mathrm{H\,c}\right) + \frac{1}{4}w(k)\,\phi^\dagger\phi\,\psi^\dagger\sigma_2\psi^{\dagger\mathrm{T}}\,\psi^{\mathrm{T}}\sigma_2\psi\right]$$

- "breakup" terms v, w allow 3-body physics to feed in properly
- 30 diagrams in evolution equation
  (automate generation!)

Initial conditions

As $k \to \infty$ boson field purely auxiliary

(bosonises 4-fermion contact interaction $G(\psi^\dagger\psi)^2$)

- Z_ϕ, λ, u_2, v, $w \to 0$
- $u_1(K)$ chosen so that in physical limit ($k \to 0$)

$$u_1(0) = -\frac{M g^2}{4\pi a}$$

a: AA scattering length

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DD scattering length

- results converge when "breakup" terms are included
- resulting value only weakly dependent on cutoff (c_B)
- stationary very close to expected "optimum" $c_B = 1$
- final result: $a_B/a \simeq 0.58 \pm 0.02$
- agrees well with full few-body result $a_B/a = 0.6$
  [Petrov, Salomon and Shlyapnikov (2004)]

Bosons

(also ≥ 3 species of fermion in spatially symmetric channels: ^{4}He)

More interesting: Efimov effect in 3-body sector

- tower of 3-body bound states, energies in geometric series
- momentum scaling factor e^{π/s_0} where $s_0 = 1.00624$
- energies in ratio ~ 515
- two 4-body bound states below each 3-body state
  [von Stecher *et al* (2009); Deltuva (2010)]

Introduce trimer field $\chi(x)$

- include energy dependence associated with 3-body bound states

Effective action

$$
\begin{aligned}
&\Gamma_k[\psi,\psi^*,\phi,\phi^*,\chi,\chi^*] \\
&= \int \mathrm{d}^4x \left[\psi^* \left(\mathrm{i}\partial_0 + \frac{\nabla^2}{2m} \right) \psi + Z_d \, \phi^* \left(\mathrm{i}\partial_0 + \frac{\nabla^2}{4m} \right) \phi + Z_t \, \chi^* \left(\mathrm{i}\partial_0 + \frac{\nabla^2}{6m} \right) \chi \right. \\
&\qquad - u_d \phi^* \phi - u_t \chi^* \chi - \frac{g}{2} (\phi^* \psi\psi + \psi^* \psi^* \phi) - h(\chi^* \phi \psi + \phi^* \psi^* \chi) \\
&\qquad - \lambda \, \phi^* \psi^* \phi \psi \\
&\qquad - \frac{u_{dd}}{2} (\phi^* \phi)^2 - \frac{v_d}{4} (\phi^* \phi^* \phi \psi \psi + \phi^* \psi^* \psi^* \phi \phi) - \frac{w}{4} \phi^* \psi^* \psi^* \phi \psi \psi \\
&\qquad - u_{tt} \, \chi^* \psi^* \chi \psi - \frac{u_{dt}}{2} (\phi^* \phi^* \chi \psi + \chi^* \psi^* \phi \phi) \\
&\qquad \left. - \frac{v_t}{2} (\phi^* \psi^* \psi^* \chi \psi + \chi^* \psi^* \phi \psi \psi) \right]
\end{aligned}
$$

AD interaction λ regenerated by evolution even if zero initially
(unlike AA scattering)

$\rightarrow$ use running trimer field [cf Gies and Wetterich (2002)]
(shift of expansion point for χ field)

$$\partial_k\chi = \zeta_1\,\phi\psi + \zeta_2\,\psi^*\chi\psi + \zeta_3\,\psi^*\phi\phi + \zeta_4\,\psi^*\phi\psi\psi$$

where $\zeta_1 = -\partial_k\lambda/2h$ to cancel running of λ

- other terms do same for four-atom couplings v_d, w and v_t
- additional piece in evolution equation

$$\partial_k\Gamma = -\frac{i}{2}\,\mathrm{Tr}\left[(\partial_k\mathbf{R})\left((\mathbf{\Gamma}^{(2)} - \mathbf{R})^{-1}\right)\right] + \frac{\delta\Gamma}{\delta\chi}\cdot\partial_k\chi$$

- 4-body equations with structure like Faddeev-Yakubovsky
(coupled DD, AT channels)

3-body sector

Coupled equations for $u_t(k)$, $Z_t(k)$ and $h^2(k)$

Scaling limit $k \gg 1/|a|$

- couplings oscillate sinusoidally with $\ln k$
- poles in AD scattering amplitude h^2/u_t
  (values of k where 3-body bound states appear at zero energy)

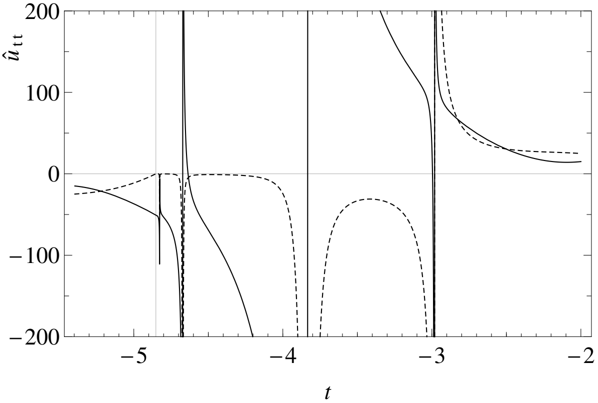
→ tower of Efimov states with $s_0 = 0.92503$ (exact: 1.006)
  [Schmidt and Moroz (2009)]
  momentum scale factor $\mathrm{e}^{\pi/s_0} = 29.2$ (exact: 22.7)

- tower cuts off when $k \sim 1/|a|$

4-body sector

3-body cycles → cyclic behaviour in 4-body system

One Efimov cycle of rescaled $\hat{u}_{tt}(k)$ as a function of $t = \ln(k/K)$



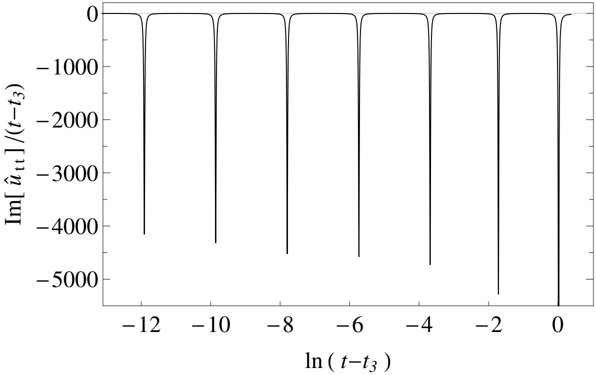
solid: real, dashed: imaginary

vertical grey line: AT threshold passes through zero energy

Comments

- imaginary part appears at AT threshold $t = t_3 \simeq -4.85$
- 4-body bound states below AT threshold $t \simeq -3.83, -4.67, \ldots$
  (decay to deeper trimer + free atom $\rightarrow$ finite widths)
- unphysical singularity from zero of $h^2(k)$ at $t \simeq -3.0$
  (end of region within cycle where $h^2(k)$, $Z_t(k)$ have opposite signs $\rightarrow$ trimer "ghost")

Infinite tower of 4-body bound states below AT threshold

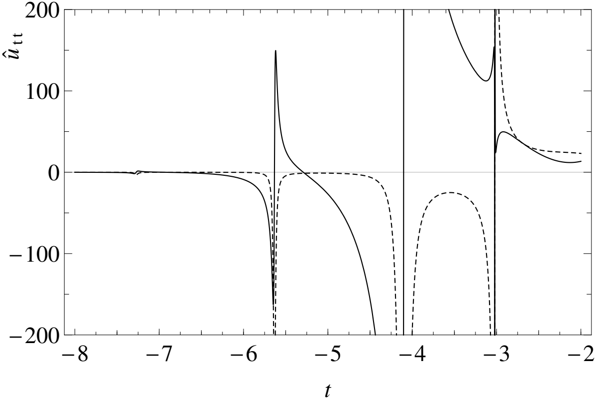


Double exponential pattern $\sim$ super-Efimov effect

[Nishida, Moroz and Son (2013)]

- but may not survive in physical limit $k \to 0$
- 4-body states move relative to AT threshold, become virtual

Final cycle of $\hat{u}_{tt}(k)$ for finite $a_0 < 0$
tuned so that last three-body state appears at $k = 0$ ($t = -\infty$)

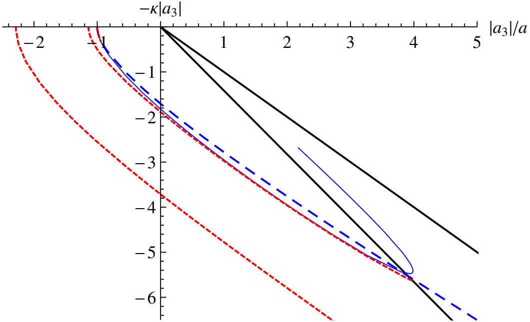


Three 4-body states, at $t = -4.1$, -5.6 and -7.1
(consistent with theorem of Amado and Greenwood)

Run at finite energy

- three 4-body bound states per cycle (one very fragile)

Plot of binding momentum $\kappa = \sqrt{-ME}$ against $1/a$



a_3: AA scattering length where trimer becomes bound (recombination);
black lines: AD, DD thresholds; blue: 3-body bound state;
red: 4-body states; thin blue: effective AT threshold

4-body states become bound at

$$a_3/a_4^{(0)} \simeq 2.29, \quad a_3/a_4^{(1)} \simeq 1.14, \quad a_3/a_4^{(2)} \simeq 1.003$$

Exact results: 2.351, 1.096 (only two states) [Deltuva (2012)]
Third state extremely weakly bound: probably artefact of truncation

Unitary limit: binding momenta

- $\kappa_3|a_3| \simeq 1.71$; cf exact: 1.5077
- $\kappa_4/\kappa_3 \simeq 2.18, 1.11$; cf exact: 2.147, 1.0011

Overall, FRG in this truncation agrees with few-body results to $\lesssim 20\%$

Summary

Applications of functional RG to 3- and 4-body systems

- local effective action, “optimised” cutoff
- keeping all local terms in 4-body sector

Fermions

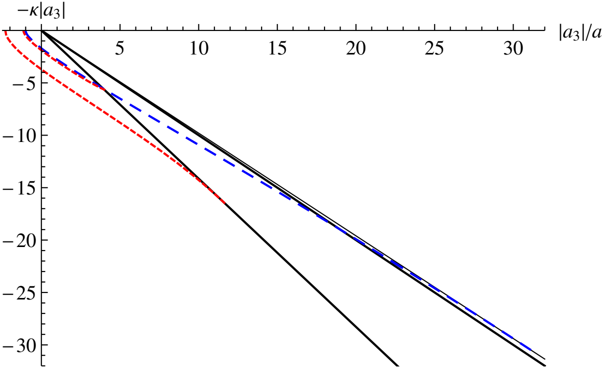
- results for dimer-dimer scattering length:
  stable against variation of cutoff, agree with direct calculations

Bosons

- dynamical trimer field (match Faddeev-Yakubovsky equations)
- 3-body sector: Efimov cycles
- 4-body sector: three states per cycle (one very weakly bound)
- energies agree with direct calculations to $\lesssim 20\%$

Extra slides

Plot of binding momentum $\kappa = \sqrt{-ME}$ against $1/a$



Super-Efimov effect

Relies on being close to fixed point with complex scaling
Example for fewer-body coupling g^2 at nontrivial fixed point

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a g^4 + b g^2 v + c v^2$$

with $b^2 - 4ac < 0 \rightarrow$ imaginary scaling dimension
Now consider g^2 marginal: $g^2 = g_0^2/t$ with $t = \ln(k/k_0)$
and define $\hat{v} = t v$

$$t \frac{\mathrm{d}\hat{v}}{\mathrm{d}t} = a g_0^4 + (1 + b g_0^2)\hat{v} + c \hat{v}^2$$

$\rightarrow$ cyclic behaviour in $\ln t = \ln(\ln(k/k_0))$ if

$$\left(\frac{1}{g_0^2} + b\right)^2 - 4ac < 0$$

4 bosons – close to AAD Efimov cycle [Deltuva (2012)]