The renormalisation group for nuclear forces

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Birse, McGovern and Richardson, hep-ph/9807302
Barford and Birse, hep-ph/0206146
Birse and McGovern, nucl-th/0307050
Barford and Birse, nucl-th/0406008
Richardson, hep-ph/0008118
Barford, nucl-th/0404072

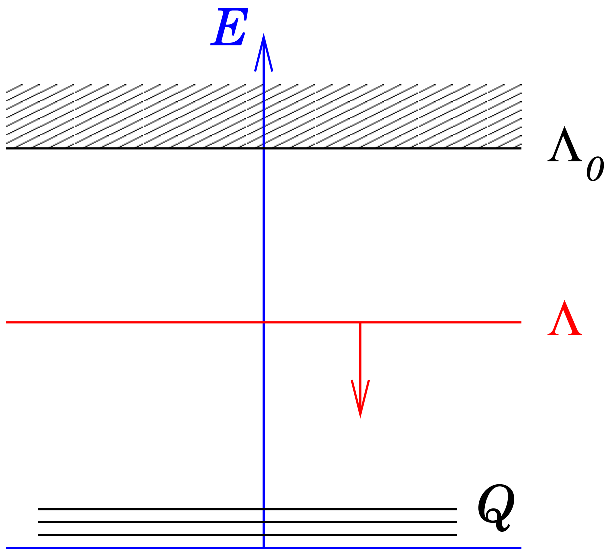
Outline

- Wilsonian renormalisation group
- RG for the nucleon-nucleon interaction
- Fixed points of short-range forces
  - Weinberg, effective-range expansion
- Energy- or momentum-dependent potentials
- Long-range forces
  - Coulomb, OPE, inverse-square
- Peripheral NN scattering
- Challenges

Wilsonian renormalisation group

Tool for analysing scale dependence of physical systems
(as developed by Wilson for condensed matter problems)

- assumes well separated scales Q (physics of interest, long distance) and Λ_0 (underlying physics, short distance)



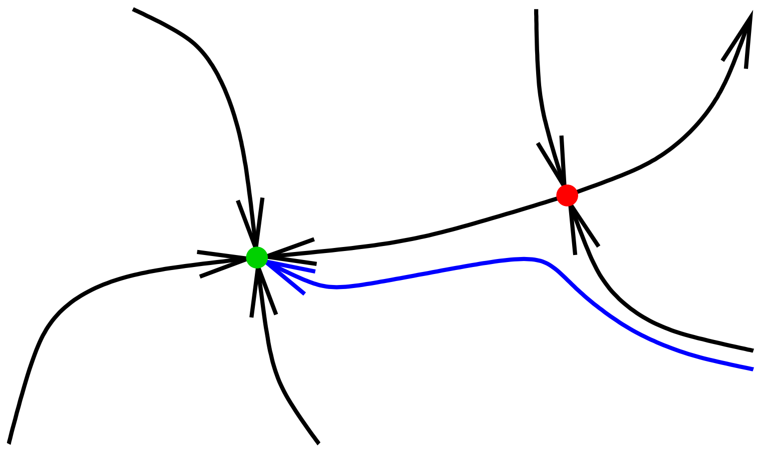
- impose floating cut-off Λ ($Q < \Lambda < \Lambda_0$)
- readjust couplings to keep observables independent of Λ
- rescale: express all dimensioned quantities in units of Λ

Follow flow in "theory space" as $\Lambda \to 0$

(space of all possible couplings for given fields and symmetries)

Look for fixed points

- rescaled theories independent of Λ
- correspond to scale-free systems
- endpoints of RG "flow"



• stable fixed point • unstable fixed point

Expand around fixed point using perturbations that scale with definite powers of Λ

- Λ^{-n} relevant or superrenormalisable (unstable)
- Λ^{n} irrelevant or nonrenormalisable
- Λ^{0} marginal or renormalisable ($\rightarrow \ln\Lambda$ scale dependence)

Resulting description of low-energy physics can be represented by an effective field theory (EFT)

- marginal terms: dimensionless couplings (as in QED, QCD)
- irrelevant terms: couplings $\propto \Lambda_0^{-n}$
  $\rightarrow$ effects suppressed by powers of $\frac{Q}{\Lambda_0}$ (as in ChPT)
- relevant terms: couplings $\propto \Lambda_0^{n}$ (masses in QFT's)

RG for the nucleon-nucleon interaction

Consider scattering by short-range potential V_S (unresolved physics) and known long-range potential V_L (eg Coulomb or OPE)
– want systematic parametrisation—power counting—for V_S

Start from Lippmann-Schwinger equation for 2-body T-matrix
– describes scattering by V_S between distorted waves (DW's) of V_L

$$\tilde{T}_S = V_S + V_S\, G_L(E)\, \tilde{T}_S$$

where G_L is the DW Green's function $G_L(E) = [E - H_0 - V_L + i\epsilon]^{-1}$

On-shell DW T-matrix elements are

$$\langle \psi_L^-(p)|\tilde{T}_S(p)|\psi_L^+(p)\rangle = -\frac{2\pi}{M_{\text{red}}}\, e^{2i\delta_L(p)}\, \frac{1}{p\left(\cot\tilde{\delta}_S(p) - i\right)}$$

in terms of additional phase shift produced by V_S

$$\tilde{\delta}_S(p) = \delta(p) - \delta_L(p)$$

Here $p = \sqrt{2M_{\text{red}}E}$ is on-shell momentum

- Impose cutoff $q \leq \Lambda$ on G_L in DW basis:

$$G_L = \frac{M_{\text{red}}}{\pi^2} \int_0^{\Lambda} \mathrm{d}q\, q^2 \, \frac{|\psi_L(q)\rangle\langle\psi_L(q)|}{p^2 - q^2} \text{ (+bound states)}$$

- Demand that fully off-shell T-(or K-)matrix be independent of Λ
  (all observables independent of Λ)

  $\rightarrow$ V_S must depend on Λ according to

$$\frac{\partial V_S}{\partial \Lambda} = -V_S \frac{\partial G_L}{\partial \Lambda} V_S$$

- Rescale all dimensioned quantities in units of cutoff Λ
  - energy, momenta: $\widehat{p} = p/\Lambda$, $\widehat{k} = k/\Lambda$
  - crucial to identify all low-energy scales in V_L, generically κ
    examples: m_π for OPE, inverse Bohr radius αM_{red} for Coulomb
  - rescale these: $\widehat{\kappa} = \kappa/\Lambda$

Also rescale short-range potential

- depends on behaviour of DW's near origin
- controlled by singularity of V_L as $r \to 0$
- if no worse than r^{-1} then define

$$\widehat{V}_S(\widehat{p}, \ldots; \Lambda) = \frac{M_{\text{red}}\Lambda}{\pi^2} V_S(\Lambda\widehat{p}, \ldots; \Lambda)$$

(covers $V_L = 0$, Coulomb, spin-singlet OPE)

Look first at s-wave scattering by pure short-range interaction, $V_L = 0$
(NN scattering at very low momenta, $p < m_\pi$)

Take potential to be δ-function plus derivatives

- with energy-dependent coefficients (more later)

$\to$ function of k^2, k'^2, p^2 in momentum space: $V_S(p, k', k; \Lambda)$

k, k' initial and final off-shell momenta (p on-shell)

RG equation

$$\Lambda \frac{\partial \widehat{V}_S}{\partial \Lambda} = \widehat{p}\,\frac{\partial \widehat{V}_S}{\partial \widehat{p}} + \widehat{k}'\,\frac{\partial \widehat{V}_S}{\partial \widehat{k}'} + \widehat{k}\,\frac{\partial \widehat{V}_S}{\partial \widehat{k}} + \widehat{V}_S + \widehat{V}_S(\widehat{k}', 1, \widehat{p}; \Lambda)\,\frac{1}{1-\widehat{p}^2}\,\widehat{V}_S(1, \widehat{k}, \widehat{p}; \Lambda)$$

Boundary conditions:

V_S should be an analytic function of k^2, k'^2 and energy (p^2)

Fixed points: solutions of RG equation that are independent of Λ

$\rightarrow$ satisfy

$$\widehat{p}\,\frac{\partial \widehat{V}_{S0}}{\partial \widehat{p}} + \widehat{V}_{S0}(\widehat{p}) + \widehat{V}_{S0}(\widehat{p})\,\frac{1}{1-\widehat{p}^2}\,\widehat{V}_{S0}(\widehat{p}) = 0$$

(assuming they depend on energy but not momentum)

Fixed points of short-range forces

Trivial fixed point: $\hat{V}_{S0}(\hat{p}) = 0$
– system with no scattering (scale free)

Real systems with weak scattering at low energies
- – describe using perturbations that scale with definite powers of Λ
- – satisfy linearised version of RG equation

$\rightarrow$ solutions which are well-behaved at low momenta

$$\hat{V}_S(\hat{p}, \hat{k}', \hat{k}; \Lambda) = C\Lambda^\nu \, \hat{k}'^{2l} \, \hat{k}^{2m} \, \hat{p}^{2n}$$

with RG eigenvalues $\nu = 2(l + m + n) + 1$ where $l, m, n \geq 0$

Eigenvalues just count powers of low-energy scales Q
- – terms of order Q^d with $d = \nu - 1$

$\rightarrow$ Weinberg power counting (as in χPT)

NN s-waves: strong scattering at low energies $\rightarrow$ nontrivial fixed point

– $1/V_{S0}$ satisifes linear equation

$$\hat{p}\,\frac{\partial}{\partial\hat{p}}\left(\frac{1}{\hat{V}_{S0}}\right)-\frac{1}{\hat{V}_{S0}(\hat{p})}-\frac{1}{1-\hat{p}^2}=0$$

– solution exactly cancels loop integral in LS equation

$\rightarrow$ infinite K-matrix: system with zero-energy bound state (scale free)

Perturbations around fixed point (energy-dependent only)

$$\frac{1}{\hat{V}_S(\hat{p},\Lambda)}=\frac{1}{\hat{V}_{S0}(\hat{p})}-\sum_{n=0}^{\infty}C_{2n}\Lambda^{2n-1}\,\hat{p}^{2n}$$

– RG eigenvalues $\nu=2n-1=-1,1,3,\ldots$ ($\nu=-1\rightarrow$ unstable)

– KSW power counting: order Q^d with $d=\nu-1=2n-2$

– one-to-one correspondance with terms in effective-range expansion

$$\frac{1}{K(p)}=-\frac{M_{\rm red}}{2\pi}\left(-\frac{1}{a}+\frac{1}{2}r_e\,p^2+\cdots\right)=-\frac{M_{\rm red}}{\pi^2}\sum_{n=0}^{\infty}C_{2n}\,p^{2n}$$

Energy- or momentum-dependent potentials

Perturbations around nontrivial fixed point include

$$\delta\widehat{V}_S(\widehat{k}',\widehat{k},\widehat{p}) = \Lambda^{2n}\left[\widehat{k}^{2n} - \widehat{p}^{2n} + \sum_{m=0}^{n-1}\frac{\widehat{p}^{2m}}{2n-2m+1}\widehat{V}_{S0}(\widehat{p})\right]\widehat{V}_{S0}(\widehat{p})$$

and ones with similar factors involving k'

– RG eigenvalues $\nu = 2n = 2, 4, 6\ldots$

$\rightarrow$ momentum-dependent terms less relevant than energy-dependent

Trivial fixed point: k^2, k'^2, p^2 all of same order

– can make transformation ("use equations of motion")

– swap momentum for energy dependence – same power counting

Energy-dependent potential $\sim$ (stepwise) Bloch-Horowitz reduction of Hilbert space to smaller "model space"

$$V_S(\Lambda - \Delta\Lambda) = V_S(\Lambda) + V_S(\Lambda)\frac{Q(\Lambda)}{E - Q(\Lambda)H_0Q(\Lambda)}V_S(\Lambda)$$

$Q(\Lambda)$ projects onto eliminated states $\Lambda - \Delta\Lambda < q < \Lambda$

Alternative: Lee-Suzuki similarity transformation

- keeps potential energy-independent
- used by Kuo and coworkers to generate effective potential $V_{\text{low}-\text{k}}$

Preserving half-off-shell T-matrix $T(p,k,p)$

$\rightarrow$ evolution equation

$$\frac{\partial}{\partial\Lambda}V_{\text{low}-\text{k}}(k',k;\Lambda) = \frac{M_{\text{red}}}{\pi^2}\,V_{\text{low}-\text{k}}(k',\Lambda;\Lambda)\,\frac{1}{1-(k/\Lambda)^2}\,T(\Lambda,k,\Lambda)$$

[Bogner et al, nucl-th/0108041, nucl-th/0305035]

Could use field transformation or folded diagrams

- eliminate energy-dependence from $V_S(p;\Lambda) \rightarrow$ "$V_{\text{very}-\text{low}-\text{k}}(k',k;\Lambda)$"
- only input: coefficients from effective-range expansion
- but unnatural coefficients for momentum-dependent perturbations

$\rightarrow$ power counting not obvious (more complicated evolution)

Long-range forces

Coulomb potential $V_L(r) = \alpha/r$

- extra low-energy scale for small α: $\kappa = \alpha M_{\text{red}}$
- potential of order Q^{-1} (like fixed point) $\rightarrow$ resum to all orders

s-wave DW's at origin have form (Sommerfeld)

$$|\psi_L(p,0)|^2 = \mathcal{C}(\kappa/p) \equiv \frac{2\pi\kappa/p}{e^{2\pi\kappa/p}-1}$$

- rescale V_S as above $\rightarrow$ RG equation (energy dependent)

$$\Lambda\frac{\partial}{\partial\Lambda}\left(\frac{1}{\widehat{V}_S}\right) = \widehat{p}\frac{\partial}{\partial\widehat{p}}\left(\frac{1}{\widehat{V}_S}\right) + \widehat{\kappa}\frac{\partial}{\partial\widehat{\kappa}}\left(\frac{1}{\widehat{V}_S}\right) - \frac{1}{\widehat{V}_S} - \frac{C(\widehat{\kappa})}{1-\widehat{p}^2}$$

Trivial fixed point $\rightarrow$ perturbative expansion in powers of p^2, κ

Nontrivial fixed point: $1/V_{S0}$ cancels analytic parts of LS loop integral

$\rightarrow$ terms in potential correspond to DW effective-range expansion

[Kong and Ravndal, hep-ph/9903523]

One-pion exchange

- low-energy scale $m_\pi = 140$ MeV
- also $\lambda_\pi = \dfrac{16\pi f_\pi^2}{M_N g_A^2} \simeq 300$ MeV $\to$ high-energy or low?

Two choices:

- λ_π built out of QCD scales M_N, f_π $\to$ high energy (χPT)
  $\to$ potential of order Q^0 (like effective-range term)
  - KSW scheme: treat OPE perturbatively
    [Kaplan, Savage and Wise, nucl-th/9802075]
  - but expansion converges at best slowly
    [Fleming, Mehen and Stewart, nucl-th/9911001]

- λ_π only $\sim 2m_\pi$ $\to$ low-energy
  $\to$ treat λ_π as new low-energy scale
  $\to$ potential of order Q^{-1} – part of fixed point
  - Weinberg–van Kolck scheme: iterate OPE to all orders
    [van Kolck, nucl-th/9902015]

Weinberg–van Kolck scheme

Spin-singlet channels: leading-order OPE just Yukawa

$$V_L(r) = -\frac{m_\pi^2}{M_N \lambda_\pi}\frac{e^{-m_\pi r}}{r}$$

RG analysis similar to Coulomb
– but with two low-energy scales: m_π and $\kappa_\pi = m_\pi^2/2\lambda_\pi$ (both $\mathcal{O}(Q)$)

Nontrivial fixed point $\rightarrow$ DW effective-range expansion

$$\begin{aligned}\mathcal{C}_\pi(\kappa_\pi/p, m_\pi/p)\cot\delta_S \;&=\; 2\kappa_\pi\Big(\mathcal{H}_\pi(\kappa_\pi/p, m_\pi/p) + \ln\kappa_\pi/\mu\Big)\\ &\quad + \frac{2}{\pi}\sum_{l,m,n} C_{lmn}\, m_\pi^{2l}\,\kappa_\pi^m\, p^{2n}\end{aligned}$$

– all nonanalytic behaviour contained in $\mathcal{C}_\pi$, $\mathcal{H}_\pi$ and $\ln\kappa_\pi$
(long-range parts of loop integral over DW's)

But connection to χPT lost if λ_π treated as low-energy scale

Inverse-square potential

Centrifugal potential ($L > 0$ waves) and three-body systems

$$V_L(r) = \frac{L(L+1)}{2M_{\text{red}}r^2}$$

DW's vanish as $r \to 0$

$$|\psi_L(p,r)|^2 \sim \frac{\pi}{4\Gamma(L+3/2)^2}\left(\frac{pr}{2}\right)^{2L} \equiv |\mathcal{N}|^2(pr)^{2L}$$

$\to$ V_S either $2L$-th derivative of δ-function (integer L)
– or small but nonzero range, such as δ-shell

$$V_S(r) = V_S(p;\Lambda)\,\frac{\delta(r-R)}{4\pi R^2}$$

R arbitrary "factorisation" scale
– separates off region of nonperturbative high-energy physics
(works for any real or complex L; for integers just numerical derivative)

Rescaled potential

$$\hat{V}_S(\hat{p};\Lambda) = \frac{M_{\rm red}}{\pi^2}\,\Lambda^{2L+1}\,R^{2L}\,V_S(\Lambda\hat{p},\Lambda)$$

satisfies RG equation

$$\Lambda\frac{\partial\hat{V}_S}{\partial\Lambda} = \hat{p}\,\frac{\partial\hat{V}_S}{\partial\hat{p}} + (2L+1)\hat{V}_S + \frac{|\mathcal{N}|^2}{1-\hat{p}^2}\,\hat{V}_S(\hat{p};\Lambda)^2$$

Trivial fixed point $\hat{V}_S = 0$ (nontrivial highly unstable)

– leading perturbation $\hat{V}_S = C\Lambda^{2L+1}$ $\to$ order Q^{2L}
(as expected for integer L: equivalent to $k'^L\,k^L$ term)

Attractive $1/r^2$ potential in spin-doublet nd channel (triton)

- complex L $\to$ wave functions oscillate as $r \to 0$
- origin of Efimov effect (tower of geometrically spaced bound states)
- need to fix self-adjoint extension of Hamiltonian
  [Bawin and Coon, quant-ph/0302199]
- leading three-body force: marginal perturbation around limit cycle
  [Bedaque et al, nucl-th/9906032; nucl-th/0207034]

Peripheral NN scattering

Use DW methods to remove effects of known pion-exchange potentials
- peripheral waves: test expansion of pion-exchange potential
- chiral OPE + TPE up to order Q^3

  [Kaiser et al, nucl-th/9706045; Rentmeester et al, nucl-th/9901054]
- OPE multiplied by $M/E \to$ order-Q^2 correction [Friar, nucl-th/9901082]

Weak scattering for large $L \to$ Weinberg power counting
- start from DW K-matrix (simpler than high-order perturbation theory)

$$\tilde{K}_S(p) = -\frac{2\pi}{M_{\text{red}}\, p} \tan\left(\delta(p) - \delta^{(0)}_{\text{OPE}}(p)\right)$$

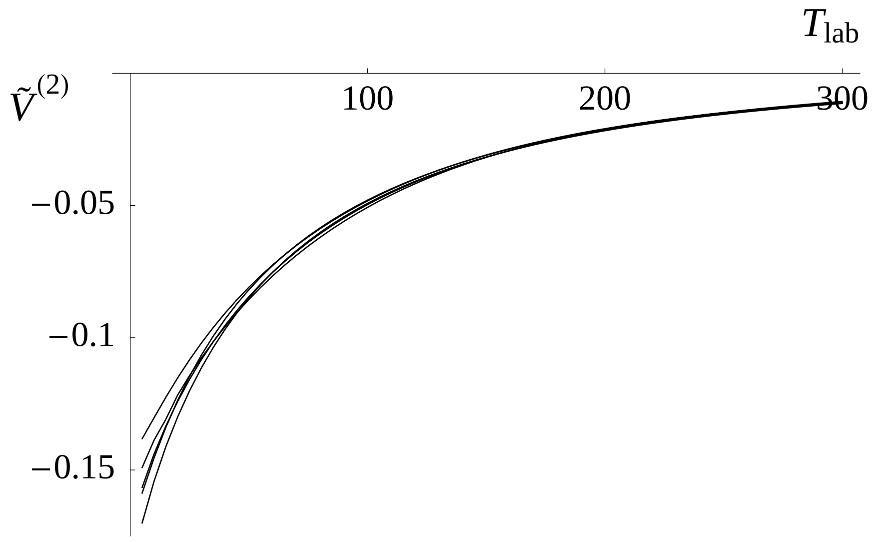
  $\delta(p)$ empirical phase shift, taken from five Nijmegen analyses (1993)
- then subtract order-Q^2 OPE and order-$Q^{2,3}$ TPE

$\to$ residual scattering starts at order Q^4: δ-shell at $r = R$ with strength

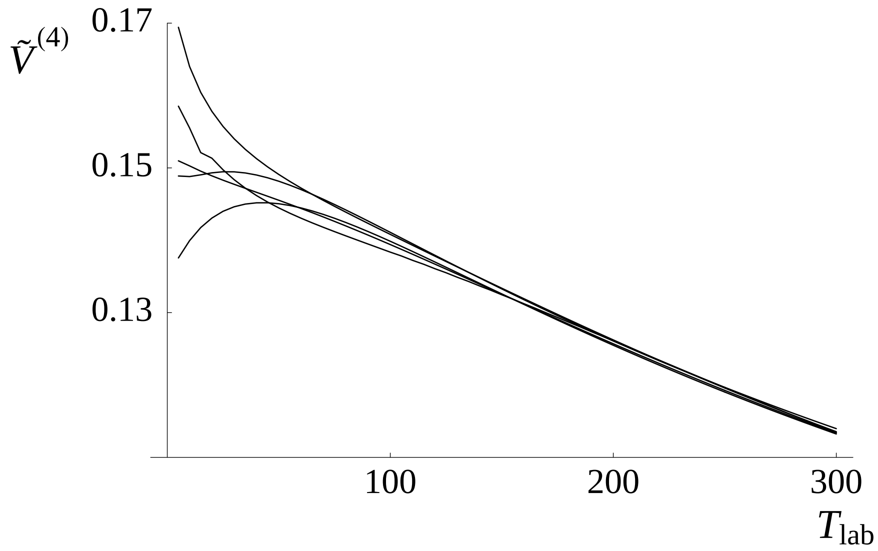
$$\tilde{V}^{(4)}(p) = \frac{R^{2L}}{[(2L+1)!!\, \psi_{\text{OPE}}(p,R)]^2} \times \left(\tilde{K}(p) - \langle\psi_{\text{OPE}}(p)|V^{(2,3)}_{\text{O,TPE}}|\psi_{\text{OPE}}(p)\rangle\right)$$

Results (strength of residual scattering)

$np\ ^1D_2$



Leading-order OPE removed

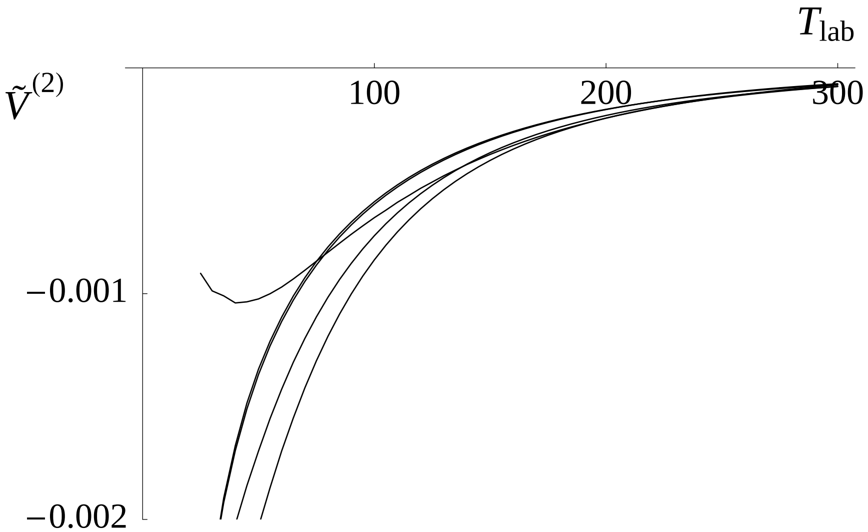


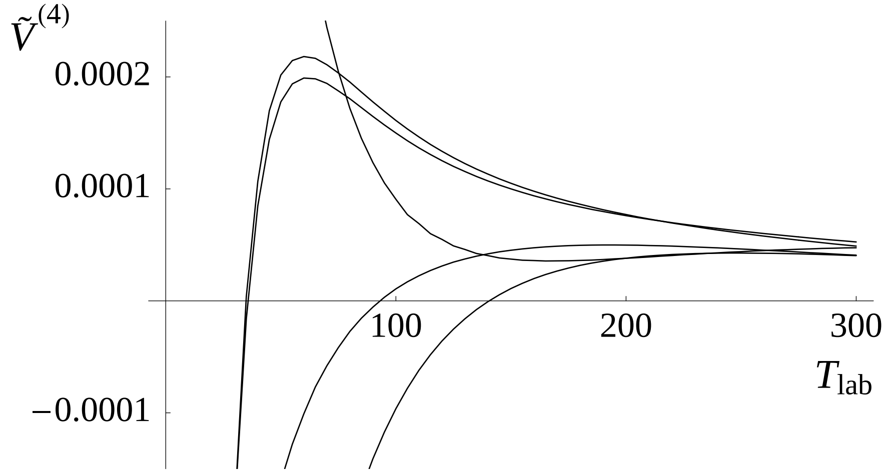
Order-$Q^{2,3}$ OPE and TPE removed

Below about 80 MeV – significant differences between PWA's

After subtraction of order-$Q^{2,3}$ terms $\sim$ linearly dependent on energy
$\rightarrow$ energy dependence well-described by chiral TPE

$np\ ^1G_4$





Leading-order OPE removed

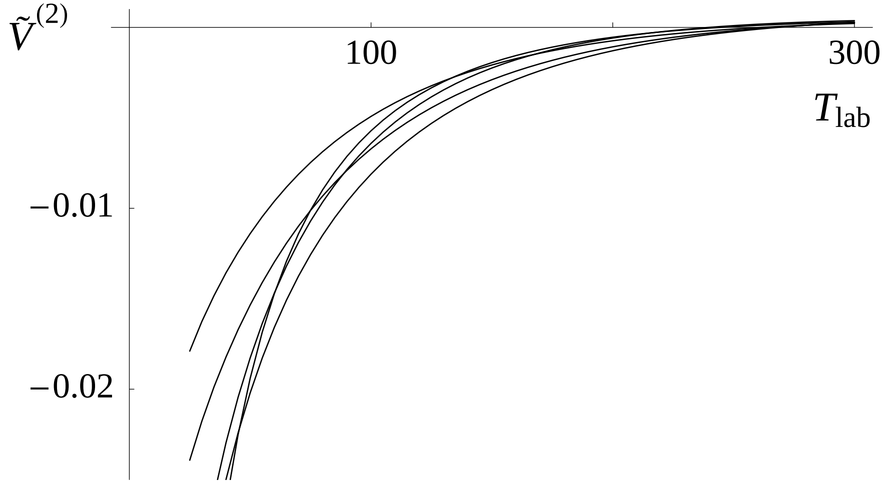
Order-$Q^{2,3}$ OPE and TPE removed

Large differences between PWA's even up to 200 MeV

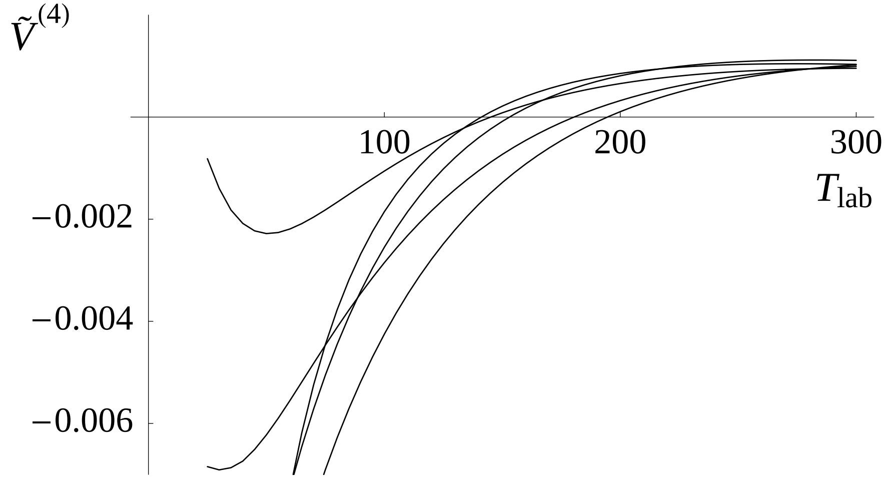
$\rightarrow$ hard to draw definite conclusions

but residual scattering much smaller after order-$Q^{2,3}$ terms subtracted

$np\ ^1F_3$



Leading-order OPE removed



Order-$Q^{2,3}$ OPE and TPE removed

Similar picture to $^1G_4$

Downward curvature at low energies for all PWA's (also in $^1P_1$)
$\rightarrow$ possible hint of isospin-breaking in πN couplings
(isospin-singlet waves – fitted to np data)

Substantial differences between the various PWA's at low energies

– $^1F_3$ and $^1G_4$ waves: artefacts completely dominate
– but no correlation among deviations $\rightarrow$ no systematic bias in fits
  (except possibly in isospin-singlet waves – CSB?)
– important to use same πN coupling as assumed in PWA
  and to include M/E factor multiplying OPE

Momentum scales of residual interactions

$$\tilde{V}^{(4)} \sim (L!)^2 \Lambda_0^{-6}\, m_\pi^{4-2L}\, k^{2L}\, g(p/m_\pi)$$

- $^1D_2$ intercept corresponds to $\Lambda_0 \sim 200$ MeV – unaturally large
- $^1D_2$ slope corresponds to $\Lambda_0 \sim 370$ MeV
- $^1F_3$, $^1G_4$ scales in range $300 - 400$ MeV

No evidence for breakdown of EFT in these peripheral waves
and no need to introduce model form factors etc
(but large systematic uncertainties in available PWA's)

Summary

Combination of renormalisation-group and distorted-wave methods:

- powerful tool for analysing low-energy interactions between nucleons
- DW's allow clean separation of known long-distance physics from unknown short-distance
- RG then gives systematic classification of terms in effective potential
- works for nonperturbative systems (where simple Weinberg power counting does not apply)

Results not new

- effective-range expansion and DW versions [Bethe, Schwinger, Blatt and Jackson, . . . , $\sim$ 1950]
- and extensions to three-body systems [Phillips, Efimov, Brayshaw, Noyes, . . . , $\sim$ 1970]
- but EFT framework $\rightarrow$ effective couplings to EM and weak currents

Challenges

- Extension to tensor OPE (spin-triplet channels)
  - $1/r^3$ form at short distances

  $\rightarrow$ must be treated nonperturbatively above critical momentum
  - ${}^3S_1$–${}^3D_1$ channels: $p \gtrsim 0.68 \dfrac{\lambda_\pi}{3} \simeq 66$ MeV (chiral limit)

- Direct determination of NN potential from empirical phase shifts and chiral pion-exchange forces
  - use DW Born and effective-range expansions
  - transform to energy-independent form $\rightarrow$ direct derivation of $V_{\mathrm{low-k}}$ (without going via model potentials)

- Use DW expansions directly in PWA's of scattering data

  $\rightarrow$ fit parameters with EFT interpretations (unlike current Nijmegen fits)

- apply effective potentials to nuclear structure
  - take effective two- and three-body potentials in vaccum renormalised at high cutoff scale
  - use as starting point for “no-core” shell model calculations initial model space: large number of oscillator shells for finite nuclei
  - then evolve down by eliminating oscillator shells using either Bloch-Horowitz [Haxton and Song, nucl-th/9907097] or Lee-Suzuki [Navrátil et al, nucl-th/9907054]
  - related example: application of “exact” RG to fermionic matter [Birse et al, hep-ph/0406249]