Pairing in many-fermion systems: an exact-renormalisation-group treatment

Michael C Birse, Judith A McGovern

(University of Manchester)

Boris Krippa, Niels R Walet

(UMIST)

Birse, Krippa, McGovern and Walet, hep-ph/0406249

ERG method:

Berges, Tetradis and Wetterich, hep-ph/0005122
Delamotte, Mouhanna and Tissier, cond-mat/0309101

Background

Ideas of effective field theory and renormalisation group now well-developed for few-nucleon systems

– rely on separation of scales

– RG can be used to derive power counting

$\rightarrow$ classify terms as perturbations around fixed point

– consistent extension of old ideas (effective-range expansion)

Many unsuccessful attempts to extend to nuclear matter

– problem: no separation of scales

– only consistent EFT so far: weakly repulsive Fermi gas
  [Hammer and Furnstahl, nucl-th/0004043]
  (reproduces old results of Bishop and others)

Other EFT's for interacting Fermi systems exist:
– Landau Fermi liquid [Shankar], Ginsburg-Landau theory
– but parameters have no simple connection to underlying forces (like ChPT and QCD)

Look for some more heuristic approach
– based on field theory
– can be matched onto EFT's for few-nucleon systems
– input from two-body (and three-body) systems in vacuum

Try "exact" renormalisation group
– based on Wilsonian RG approach to field theories
– successfully applied to various systems in particle physics and condensed-matter physics
[version due to Wetterich, Phys Lett **B301** (1993) 90]

Outline

- ERG for the effective action
- Effective action for fermions with attraction
- Choice of regulator functions
- Evolution equations: general structure
- Driving terms in evolution equations
- Initial conditions
- Results

ERG for the effective action

For single real scalar field ϕ start from

$$e^{iW[J]} = \int D\phi\, e^{i(S[\phi]+J\cdot\phi-\frac{1}{2}\phi\cdot R\cdot\phi)}$$

$R(q,k)$: regulator function for the ERG

– suppresses contributions of modes with low momenta, $q \lesssim k$

$\rightarrow$ only modes with $q \gtrsim k$ integrated out

$\rightarrow$ $W[J]$ evolves with regulator ("cut-off") scale k

– becomes full generating function as $k \rightarrow 0$

Legendre transform $\rightarrow$ effective action $\Gamma[\phi_c]$ (generator for 1PI diagrams) where expectation value of the field is

$$\frac{\delta W}{\delta J} \equiv \phi_c$$

[See talks by: Furnstahl, Litim, Polonyi]

Effective action

[convention as in: Weinberg, Quantum Theory of Fields II]

$$\Gamma[\phi_c] = W[J] - J \cdot \phi_c + \frac{1}{2}\,\phi_c \cdot R \cdot \phi_c$$

W evolves with scale k according to

$$\partial_k W = -\frac{1}{2}\,\phi_c \cdot \partial_k R \cdot \phi_c + \frac{i}{2}\,\mathrm{Tr}\left[(\partial_k R)\,\frac{\delta\phi_c}{\delta J}\right]$$

Evolution of $\Gamma[\phi_c]$

– J also runs if ϕ_c is held constant

– $\phi_c \cdot \partial_k R \cdot \phi_c$ terms cancel

$$\partial_k \Gamma = \frac{i}{2}\,\mathrm{Tr}\left[(\partial_k R)\,\frac{\delta\phi_c}{\delta J}\right]$$

From definition of Γ

$$\frac{\delta J}{\delta\phi_c} = -(\Gamma^{(2)} - R) \qquad \text{where} \qquad \Gamma^{(2)} = \frac{\delta^2\Gamma}{\delta\phi_c\delta\phi_c}$$

$\longrightarrow$ evolution equation for Γ in form of a one-loop integral

$$\partial_k\Gamma = -\frac{i}{2}\,\mathrm{Tr}\left[(\partial_k R)\,(\Gamma^{(2)} - R)^{-1}\right]$$

$(\Gamma^{(2)} - R)^{-1}$: propagator of boson in background field ϕ_c

One-loop structure: like RG for few-body systems

$\longrightarrow$ can match ERG in matter onto interactions in vacuum

[Polchinski's version of ERG: different structure

– see for example: Morris, hep-th/9802039]

For a system of fermions as well as bosons

$$\begin{aligned}\partial_k \Gamma &= +\frac{i}{2}\,\mathrm{Tr}\left[(\partial_k \mathbf{R}_F)\left((\mathbf{\Gamma}^{(2)} - \mathbf{R})^{-1}\right)_{FF}\right] \\ &\quad -\frac{i}{2}\,\mathrm{Tr}\left[(\partial_k \mathbf{R}_B)\left((\mathbf{\Gamma}^{(2)} - \mathbf{R})^{-1}\right)_{BB}\right]\end{aligned}$$

System with charged condensates (pairing $\rightarrow$ particle-hole mixing)
– write propagator for complex field as 2×2 matrix (Nambu-Gor'kov)
$\rightarrow$ factors of $\frac{1}{2}$ still present

Regulator function and ansatz for Γ

Regulator $R(q,k)$: IR cut-off on effective action Γ
– should suppress contributions of modes with $q \lesssim k$
– should give back full effective action as $k \to 0$
$\to$ $R(q,k)$ should provide large mass/energy gap for modes with $q \lesssim k$
and should vanish for $q \gg k$ and $k \to 0$

Derivative $\partial_k R(q,k)$ in ERG equation
– peaks for $q \sim k$
– tends to zero for $q \gg k$
$\to$ acts as UV cut-off on loop integrals

ERG: complicated differential equation for functional Γ
– need to choose an ansatz for effective action
– make an expansion in local terms (as in rigorous EFT's)
– use physics to guide choice

Effective action for fermions with attraction

Attractive forces between fermions → pairing [Furnstahl, Hands]
– weak attraction: Cooper pairs (BCS state) $\mu \simeq \epsilon_F$
– strong attraction: Bose-Einstein condensation (BEC) $\mu < 0$

Single species of nonrelativistic fermion: ψ (as in neutron matter)
Boson field describing correlated fermion pairs: ϕ
Finite density: chemical potential μ
Ansatz for Γ:

$$\begin{aligned}
&\Gamma[\psi, \psi^\dagger, \phi, \phi^\dagger, \mu, k] \\
&\quad = \int d^4x \left[\phi^\dagger(x) \left(Z_\phi \left(i\partial_t + 2\mu \right) + \frac{Z_m}{2m} \nabla^2 \right) \phi(x) - U(\phi, \phi^\dagger) \right. \\
&\qquad + \psi^\dagger \left(Z_\psi (i\partial_t + \mu) + \frac{Z_M}{2M} \nabla^2 \right) \psi \\
&\qquad \left. - Z_g \, g \left(\frac{i}{2} \psi^\mathsf{T} \sigma_2 \psi \phi^\dagger - \frac{i}{2} \psi^\dagger \sigma_2 \psi^{\dagger\mathsf{T}} \phi \right) \right]
\end{aligned}$$

Potential:

Bosons carry twice charge of a fermion

– couple to chemical potential μ via quadratic term

– absorb into potential

$$\bar{U} = U - 2\mu Z_\phi \phi^\dagger \phi$$

Expand potential about minimum $\phi^\dagger\phi = \rho_0$ to quadratic order:

$$\bar{U} = u_0 + u_1(\phi^\dagger\phi - \rho_0) + \frac{1}{2}u_2(\phi^\dagger\phi - \rho_0)^2$$

(one redundant parameter: ρ_0 or u_1)

In symmetric phase: $\rho_0 = 0$

In condensed phase: u_n defined at minimum $\rightarrow u_1 = 0$

In condensed phase with uniform background ϕ field:

Particles and holes mix (ψ and $\psi^\dagger$ coupled)
$\longrightarrow$ fermion spectrum with energy gap $\Delta = g|\phi|/Z_\phi$

$$E_F(\mathbf{q}) = \pm \frac{1}{Z_\psi}\sqrt{\left(\frac{Z_M}{2M}(\mathbf{q}^2 - p_F^2)\right)^2 + g^2\phi^\dagger\phi}$$

Bosons become gapless Goldstone modes
– spectrum (ϕ and $\phi^\dagger$ also coupled)

$$E_B(\mathbf{q}) = \pm \frac{1}{Z_\phi}\sqrt{\frac{Z_m}{2m}\mathbf{q}^2\left(\frac{Z_m}{2m}\mathbf{q}^2 + 2u_2\phi^\dagger\phi\right)}$$

$\longrightarrow$ superfluid state: BCS or BEC

Γ depends on cut-off scale k through running:

- coefficients in potential, u_0, u_1 (or ρ_0), u_2
- wave-function renormalisation factors, Z_ϕ, Z_ψ
- mass renormalisations, Z_M, Z_m
- coupling constant renormalisation, Z_g

To study crossover from BCS pairing to BEC
– need to work at fixed density
(otherwise can't get to negative μ for BEC)
$\rightarrow$ must allow μ to run with k

ERG becomes a set of coupled first-order ODE's

Bare theory: at starting scale $k = K$

– two-body interaction between fermions only

$$\mathcal{L}_{\text{int}} = -\frac{1}{4} C_0 \left(\psi^\dagger \sigma_2 \psi^{\dagger\mathsf{T}}\right) \left(\psi^\mathsf{T} \sigma_2 \psi\right)$$

– bosons just auxiliary fields (Hubbard-Stratonovich)

$$C_0(K) = -\frac{g(K)^2}{u_1(K)}$$

and $Z_{\phi,m}(K) \ll 1$, $u_2(K) \ll |C_0|$
(separation of C_0 arbitrary $\rightarrow$ results independent of $g(K)$)

Fermions not dressed at $k = K \rightarrow Z_\psi(K) = Z_M(K) = Z_g(K) = 1$

Here (first study):

– allow only potential (u_n, ρ_0) and Z_ϕ to run independently

– freeze $Z_\psi = Z_M = Z_g = 1$ and set $Z_m = Z_\phi$ or 1

Choice of regulator functions

Nonrelativistic systems
– carry out loop integrals over energy exactly
– regulate only integrals over three-momentum

Bosonic regulator:

$$R_B(q,k) = \frac{k^2}{2m} f(q/k)$$

where $f(x) \to 1$ as $x \to 0$ and $f(x) \to 0$ as $x \to \infty$ (and $q = |\mathbf{q}|$)

Take $R_B(q,k) \propto k^2$ for $q \lesssim k$
$\to$ large-k behaviours of integrals reflect UV divergences

Here: use smoothed step function

$$f(q/k) = \frac{1}{2\,\mathrm{erf}(1/\sigma)}\left[\mathrm{erf}\left(\frac{q+k}{k\sigma}\right) + \mathrm{erf}\left(\frac{q-k}{k\sigma}\right)\right]$$

σ: parameter controlling sharpness

Fermionic regulator:

– should be positive for particle states ($q^2/2M > \mu$)

– and negative for hole states ($q^2/2M < \mu$)

(can't just add artificial gap term – regulator must work in vacuum)

Here: use

$$R_F(q, p_F, k) = \mathrm{sgn}(q - p_\mu)\,\frac{k^2}{2M}\,f\Big(\frac{q-p_F}{k}\Big)$$

$p_\mu = \sqrt{2M\mu}$: Fermi momentum corresponding to running μ

$p_F = (3\pi^2 n)^{1/3}$: related to density n

Symmetric phase: $p_F = p_\mu$ (until $Z_{\psi,M}$ run)

Condensed phase: "Fermi surface" no longer at p_F

(not even well-defined for large gaps)

but gap in fermion spectrum → regulator no longer crucial

Evolution equations: general structure

At present level of truncation (running u_n, ρ_0 and Z_ϕ only):
– all equations obtained from effective potential for uniform ϕ field
– evolves according to

$$\partial_k \bar{U} = -\frac{1}{\mathcal{V}_4}\partial_k \Gamma$$

$\mathcal{V}_4$: volume of spacetime

Write potential in terms of $\rho = \phi^\dagger \phi$: $\bar{U}(\rho, \mu, k)$
– coefficients

$$u_n = \left.\frac{\partial^n \bar{U}}{\partial \rho^n}\right|_{\rho=\rho_0}$$

– density and wave-function renormalisation

$$n = -\left.\frac{\partial \bar{U}}{\partial \mu}\right|_{\rho=\rho_0} \qquad Z_\phi = -\frac{1}{2}\left.\frac{\partial^2 \bar{U}}{\partial \rho \partial \mu}\right|_{\rho=\rho_0}$$

All quantities defined at running minimum $\rho = \rho_0(k)$
$\longrightarrow$ extra implicit dependence on k in condensed phase
– evolution of u_n at constant μ:

$$\partial_k u_n - u_{n+1}\,\partial_k \rho_0 = \frac{\partial^n}{\partial \rho^n}\Big(\partial_k \bar{U}\Big)\Big|_{\rho=\rho_0}$$

– couples u_2 to u_3: beyond current level of truncation

Could simply set $u_3 = 0$, but can do better:
– take $u_3(k)$ from evolution with fermion loops only
(can be solved analytically)
$\longrightarrow$ approximation becomes exact if boson loops negligible

Evolution at constant density:

Running $\mu(k) \rightarrow$ further implicit dependence on k

Define total derivative

$$d_k = \partial_k + (d_k\mu)\,\frac{\partial}{\partial\mu}$$

and apply to $\partial\bar{U}/\partial\mu \rightarrow$ evolution equation for density

$$d_k n - 2Z_\phi\, d_k\rho_0 + \chi\, d_k\mu = -\frac{\partial}{\partial\mu}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=\rho_0}$$

where fermion-number susceptibility is

$$\chi = \frac{\partial^2\bar{U}}{\partial\mu^2}\Big|_{\rho=\rho_0}$$

Keep n constant $\longrightarrow$ coupled equation for ρ_0 and μ

$$-2Z_\phi\, d_k\rho_0 + \chi\, d_k\mu = -\frac{\partial}{\partial\mu}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=\rho_0}$$

Symmetric phase: driving term on RHS vanishes and $\rho_0 = 0$
$\longrightarrow$ evolution at constant n same as at constant μ
$\longrightarrow$ much simpler set of evolution equations:

$$\begin{aligned}
\partial_k u_1 &= \frac{\partial}{\partial\rho}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=0} \\
\partial_k u_2 &= \frac{\partial^2}{\partial\rho^2}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=0} \\
\partial_k Z_\phi &= -\frac{1}{2}\,\frac{\partial^2}{\partial\mu\partial\rho}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=0}
\end{aligned}$$

Condensed phase: set of equations is

$$
\begin{aligned}
-u_2\, d_k\rho_0 + 2Z_\phi\, d_k\mu &= \frac{\partial}{\partial\rho}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=\rho_0} \\
d_k u_2 - u_3\, d_k\rho_0 + 2Z'_\phi\, d_k\mu &= \frac{\partial^2}{\partial\rho^2}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=\rho_0} \\
d_k Z_\phi - Z'_\phi\, d_k\rho_0 + \frac{1}{2}\chi'\, d_k\mu &= -\frac{1}{2}\frac{\partial^2}{\partial\mu\partial\rho}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=\rho_0}
\end{aligned}
$$

Here

$$
Z'_\phi = -\frac{1}{2}\frac{\partial^3\bar{U}}{\partial\rho^2\partial\mu}\Big|_{\rho=\rho_0} \qquad \chi' = \frac{\partial^3\bar{U}}{\partial\mu^2\partial\rho}\Big|_{\rho=\rho_0}
$$

– like u_3 and χ: beyond current level of truncation

$\longrightarrow$ replace by expressions from fermion loops only

Driving terms in evolution equations

Right-hand sides of evolution equations
– all obtained from $\partial_k \bar{U}$: sum of fermion and boson one-loop integrals

Treat ψ and $\psi^\dagger$ as independent fields (also ϕ and $\phi^\dagger$)
$\rightarrow$ 2×2 matrix structure for Gor'kov propagators etc

In presence of uniform ϕ field
– inverse fermion propagator

$$\Gamma^{(2)}_{FF} - \mathbf{R}_F = \begin{pmatrix} Z_\psi q_0 - E_{FR} + i\epsilon\,\mathrm{sgn}(q - p_\mu) & ig\phi\sigma_2 \\ -ig\phi^\dagger\sigma_2 & Z_\psi q_0 + E_{FR} - i\epsilon\,\mathrm{sgn}(q - p_\mu) \end{pmatrix}$$

where

$$E_{FR}(q, p_F, k) = \frac{1}{2M}\,q^2 - \mu + R_F(q, p_F, k)\ \mathrm{sgn}(q - p_\mu)$$

Inverse boson propagtor

$$\mathbf{\Gamma}^{(2)}_{BB} - \mathbf{R}_B = \begin{pmatrix} Z_\phi q_0 - E_{BR} + i\epsilon & -u_2\phi\phi \\ -u_2\phi^\dagger\phi^\dagger & -Z_\phi q_0 - E_{BR} + i\epsilon \end{pmatrix}$$

where

$$E_{BR}(q,k) = \frac{Z_m}{2m}q^2 + u_1 + u_2(2\phi^\dagger\phi - \rho_0) + R_B(q,k)$$

Evaluating the loop integrals gives (after some work)

$$\partial_k \bar{U} = -\frac{1}{\mathcal{V}_4}\partial_k\Gamma \;=\; -\int \frac{d^3\vec{q}}{(2\pi)^3}\frac{E_{FR}}{\sqrt{E_{FR}^2+\Delta^2}}\,\mathrm{sgn}(q-p_\mu)\,\partial_k R_F + \frac{1}{2Z_\phi}\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{E_{BR}}{\sqrt{E_{BR}^2 - V_B^2}}\partial_k R_B$$

where $\Delta^2 = g^2\phi^\dagger\phi$ and $V_B = u_2\phi^\dagger\phi$

Derivatives of $\partial_k\bar{U}$ with respect to $\rho = \phi^\dagger\phi$ and $\mu \rightarrow$ driving terms

Initial conditions

Run evolution down to $k = 0$ starting from some large scale $k = K$

Initial conditions obtained by matching onto evolution in vacuum for $k \geq K$
– fermion loops only
– can be integrated analytically to get

$$\frac{u_1(K)}{g^2} = -\frac{M}{4\pi a} + \frac{1}{2}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\left[\frac{1}{E_{FR}(q,0,0)} - \frac{1}{E_{FR}(q,0,K)}\right]$$

since $u_1(0)$ related to physical scattering amplitude at threshold by

$$T_F = \frac{4\pi a}{M} = -\frac{g^2}{u_1(0)}$$ a: scattering length

Both integrals diverge linearly on their own
– usual linear divergence in EFT's for two-body scattering
$\rightarrow$ difference linear in K
(chose $R_F \propto k^2$ for large $k \rightarrow K \sim$ cut-off scale)

Need to be careful in matter: regulator shifted by Fermi surface

– acts like cut-off at $K + p_F$ for large $K \gg p_F$

$\rightarrow$ constant shift $\propto p_F$ in linearly divergent integral

$\rightarrow$ define $u_1(K)$ for use in matter by

$$\frac{u_1(K)}{g^2} = -\frac{M}{4\pi a} + \frac{1}{2}\int \frac{d^3\vec{q}}{(2\pi)^3}\left[\frac{1}{E_{FR}(q,0,0)} - \frac{\text{sgn}(q - p_F)}{E_{FR}(q,p_F,K)}\right]$$

– like using regulator that interpolates smoothly between $R_F(q, p_F, k)$ for $k \gg p_F$ and $R_F(q, 0, k)$ for $k \lesssim p_F$

(Fermi sea in second term $\sim$ totally suppressed for $K \gg p_F$)

Other initial values: $u_2(K)$, $Z_\phi(K)$ determined similarly

(but p_F-dependence suppressed by powers of p_F/K)

Results

Technique can be applied to many systems
– for definiteness start with parameters relevant to neutron matter:
$M = 4.76$ fm^{-1}, $p_F = 1.37$ fm^{-1}, $|a| \gg 1$ fm
– then explore wider range of values for $p_F a$

Results are independent of K for $K \gtrsim 4p_F$
(provided we are careful to use shifted cut-off to define $u_1(K)$)
Also independent of width parameter σ in regulator functions

Compare results with simpler approximation keeping fermion loops only
– mean-field approximation for bosons
$\rightarrow$ analytic results
[Marani, Pistolesi and Strinati, cond-mat/9703160
Papenbrock and Bertsch, nucl-th/9811077
Babaev, cond-mat/0010085]

Mean-field effective potential (at $k = 0$)

$$U^{\mathrm{MF}}(\Delta, \mu) = \frac{{k_\Delta}^5}{2M\pi}\left[\frac{1}{8ak_\Delta} - \frac{1}{15}\left(1 + x^2\right)^{\frac{3}{4}} P^1_{\frac{3}{2}}\left(-\frac{x}{\sqrt{1 + x^2}}\right)\right]$$

$P_l^m(y)$: associated Legendre function

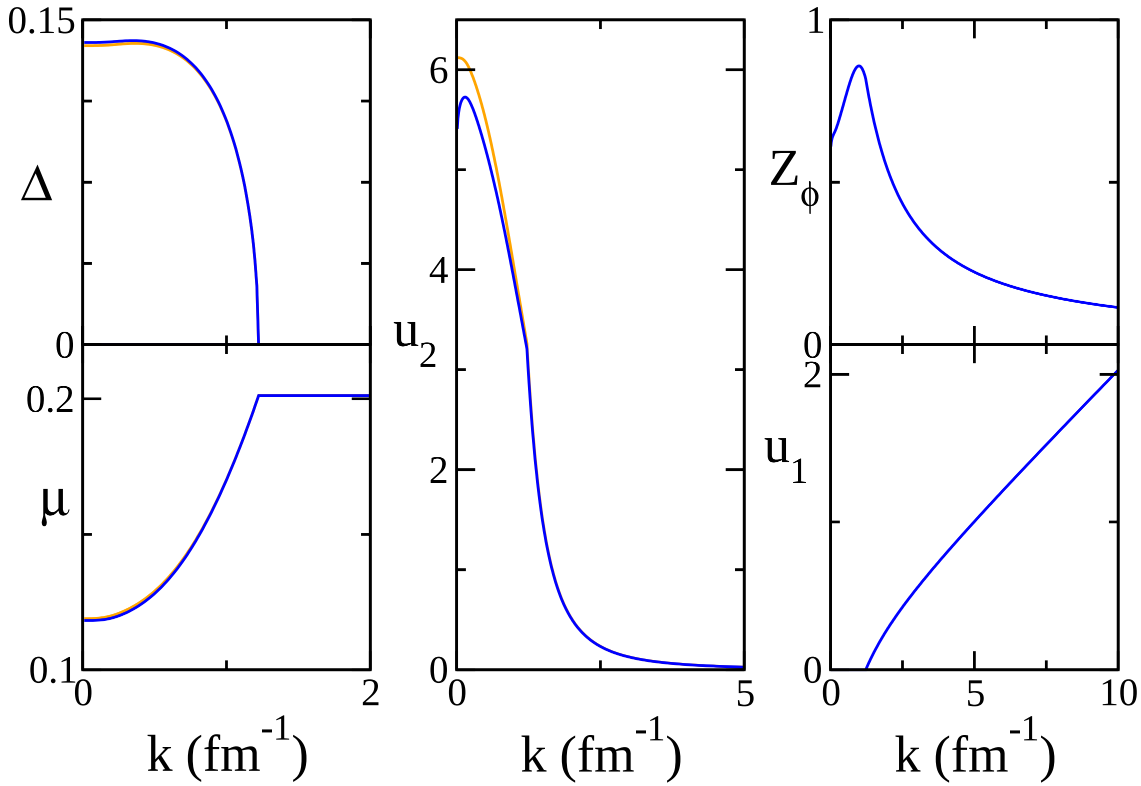
$k_\Delta = \sqrt{2M\Delta}$, $x = \mu/\Delta$, in terms of gap $\Delta = g|\phi|$

Minimise with respect to Δ at constant density

$\longrightarrow$ nonlinear equations for Δ, μ

In limit of weak attraction, $p_F a \to 0^-$, gap has exponential form

$$\Delta \simeq \frac{8}{e^2}\,\epsilon_F \exp\left(-\frac{\pi}{2p_F|a|}\right)$$

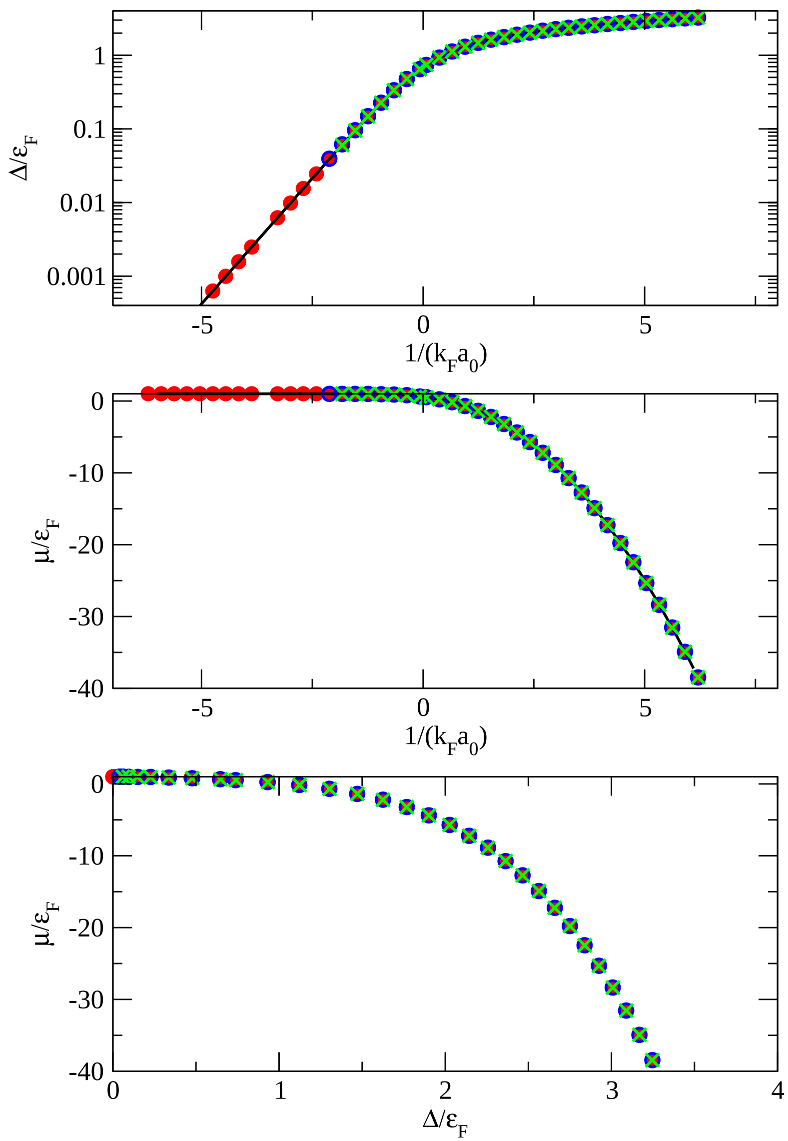


Numerical solutions to the evolution equations for infinite a_0, starting from $K = 16\ \text{fm}^{-1}$ (all in appropriate powers of fm^{-1})

—: full solution —: fermion loops only

Transition to condensed phase ($u_1 = 0$) at $k_{\text{crit}} \simeq 1.2\ \text{fm}^{-1}$
Contributions of boson loops small (negligible in symmetric phase)

Crossover from BCS to BEC



Gap Δ and chemical potential μ at physical point $(k = 0)$

– over wide range of densities or couplings, $(p_F a)^{-1}$

–: fermions only (analytical)

●: fermions only (numerical)

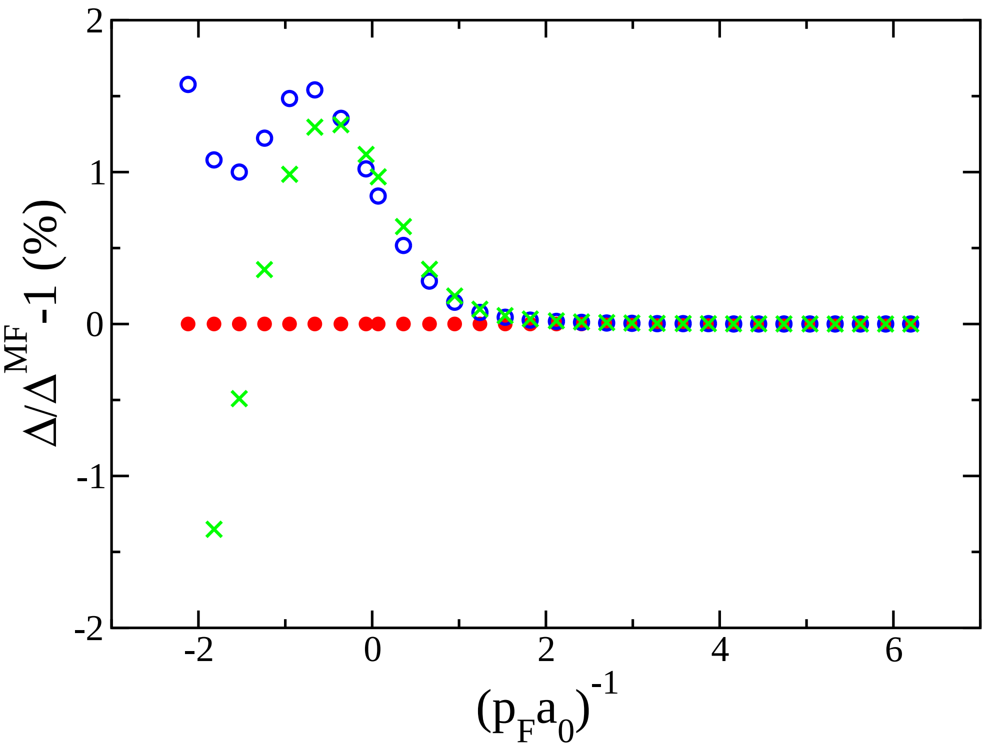
○: bosonic loops with $Z_\phi = 1$

×: full results

BCS: positive $\mu \sim p_F^2/2M$ (large negative $(p_F a)^{-1}$)

BEC: large negative μ (large positive $(p_F a)^{-1}$)

Closer look at gap



Fractional deviation of gap from analytic mean-field result

●: fermions only (numerical)

○: bosonic loops, $Z_\phi = 1$

$\times$: full results

Comments

In region of strong attraction

– contributions of boson loops to gap Δ very small

– $\sim 1\%$ enhancement of gap in large-$p_F a$ region

– effects on other quantities larger ($\sim 10\%$ in u_2)

– tend to cancel in Δ

But increasingly important for weaker couplings or lower densities

Not able to get results for $1/(p_F a_0) \lesssim -2$

– effective potential nonanalytic in ϕ for small gaps

$\longrightarrow$ expansion of effective action breaks down

For parameters corresponding to neutron matter

– gap comparable to ϵ_F (~ 30 MeV)

– more realistic treatments give $\Delta \sim 5$ MeV

$\longrightarrow$ need to keep higher-order terms in effective-range expansion

Future work

Long list of "things to do" including:

Renormalisation of boson kinetic mass, Z_m
$\longrightarrow$ scaling analysis of boson loops for small gaps

Complete analysis of current ansatz for Γ
– running of fermion renormalisation factors, $Z_{\psi,M}$
and "Yukawa" coupling, Z_g

Adding momentum-dependent interactions (effective range)
$\longrightarrow$ more realistic interaction strength at Fermi surface

Treating explicitly particle-hole channels (RPA phonons)
– important physics [Schwenk]
– remove Fierz ambiguity in bosonisation
[Jaeckel and Wetterich, hep-ph/0207094]

Adding three-body forces