Revision of Angular Momentum in Quantum Mechanics

This document summarises the aspects of angular momentum that you met in the second year courses PHYS20101 and PHYS20235.

Orbital angular momentum

We start with the classical definition of orbital angular momentum. In quantum mechanics the position and momentum vectors become operators, so

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \Rightarrow \widehat{L}_z = -i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) = -i\hbar\frac{\partial}{\partial \phi} \quad \text{etc}$$
$$[\widehat{L}_x, \widehat{L}_y] = i\hbar\widehat{L}_z \quad \text{etc;} \qquad [\widehat{\mathbf{L}}^2, \widehat{L}_i] = 0;$$

The commutation relations imply that we can only simultaneously know $\mathbf{L}^2$ and one component, taken conventionally to be L_z. The common eigenfunctions of $\widehat{\mathbf{L}}^2$ and $\widehat{L}_z$ are the spherical harmonics, $Y_l^m(\theta, \phi)$:

$$\widehat{\mathbf{L}}^2 Y_l^m(\theta,\phi) = \hbar^2\, l(l+1)\, Y_l^m(\theta,\phi) \qquad \widehat{L}_z Y_l^m(\theta,\phi) = \hbar m\, Y_l^m(\theta,\phi)$$

From requirements that the wave function must be finite everywhere, and single-valued under $\phi \to \phi + 2\pi$, it emerges that l and m are integers and must satisfy

$$l = 0, 1, 2 \ldots, \qquad m = -l, -l+1, \ldots l.$$

These have definite parity of $(-1)^l$, since under $\mathbf{r} \to -\mathbf{r}$,

$$Y_l^m(\theta,\phi) \to Y_l^m(\pi - \theta, \phi + \pi) = (-1)^l Y_l^m(\theta,\phi).$$

See the end of these notes for some explicit forms of spherical harmonics.

Intrinsic and total angular momentum

Orbital angular momentum is not the only source of angular momentum, particles may have *intrinsic angular momentum or spin*. The corresponding operator is $\widehat{\mathbf{S}}$. The eigenvalues of $\widehat{\mathbf{S}}^2$ have the same form as in the orbital case, $\hbar^2 s(s+1)$, but now s can be integer or half integer; similarly the eigenvalues of $\widehat{S}_z$ are $\hbar m_s$, with

$$s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2} \ldots, \qquad m_s = -s, -s+1, \ldots s.$$

$s = \frac{1}{2}$ for an electron, $s = 1$ for a photon or W boson. This means that the magnitude of the spin vector of an electron is $(\sqrt{3}/2)\hbar$, but we always just say "spin-$\frac{1}{2}$".

If a particle has both orbital and spin angular momentum, we talk about its *total angular momentum*, with operator

$$\widehat{\mathbf{J}} = \widehat{\mathbf{L}} + \widehat{\mathbf{S}}.$$

As with spin, the eigenvalues of $\widehat{\mathbf{J}}^2$ are $\hbar^2 j(j+1)$,

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2} \ldots, \qquad m_j = -j, -j+1, \ldots j.$$

Systems composed of more than one particle (hadrons, nuclei, atoms) will have many contributions to their total angular momentum. It is sometimes useful to add up all the spins to give a total spin, and now, confusingly, we denote the quantum numbers by S and M_S, so it is really important to distinguish operators and the corresponding quantum numbers. Then

$$\widehat{\mathbf{S}}^{\rm tot} = \widehat{\mathbf{S}}^{(1)} + \widehat{\mathbf{S}}^{(2)} + \ldots,$$

where the superscripts (1), (2) refer to the individual particles.

Similarly we use $\widehat{\mathbf{L}}^{\rm tot}$ with quantum numbers L and M_L, and $\widehat{\mathbf{J}}^{\rm tot}$ with quantum numbers J and M_J. When talking about angular momentum generally, we often use $\widehat{\mathbf{J}}$ to refer to *any* angular momentum, whether single or multiple particle, pure spin, pure orbital or a combination.

The following rules are obeyed by *any* angular momentum (eg $\widehat{\mathbf{J}}$ can be replaced by $\widehat{\mathbf{L}}$ or $\widehat{\mathbf{S}}$, for a single particle of composite system):

$$[\widehat{J}_x, \widehat{J}_y] = i\hbar \widehat{J}_z \quad \text{etc;} \qquad [\widehat{\mathbf{J}}^2, \widehat{J}_i] = 0;$$

It follows that the eigenvalues of $(\widehat{\mathbf{L}}^{\rm tot})^2$, $(\widehat{\mathbf{S}}^{\rm tot})^2$ and $(\widehat{\mathbf{J}}^{\rm tot})^2$ have exactly the same form, with the same restrictions on the quantum numbers, as those for a single particle. So for instance the eigenstates of $(\widehat{\mathbf{S}}^{\rm tot})^2$ are $\hbar^2 S(S+1)$, and of $\widehat{S}_z^{\rm tot}$ are $\hbar M_s$, and

$$\begin{array}{ccc} L = 0, 1, 2 \ldots, & S = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots, & J = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots, \\ M_L = -L, -L+1, \ldots L, & M_S = -S, -S+1, \ldots S, & M_J = -J, -J+1, \ldots J. \end{array}$$

Addition of angular momentum

The rules for the addition of angular momentum are as follows: we start with adding orbital angular momentum and spin for a composite system with quantum numbers L and S. Angular momentum is a vector, and so the total can be smaller as well as greater that the parts; however the z-components just add. The allowed values of the total angular momentum quantum numbers are

$$J = |L - S|, |L - S| + 1, \ldots, L + S, \qquad M_J = M_L + M_S.$$

However since $\widehat{L}_z$ and $\widehat{S}_z$ do not commute with $\widehat{\mathbf{J}}^2$, we cannot know J, M_L and M_S simultaneously. For a single-particle system, replace J, L, and S with j, l, and s.

More generally, for the addition of any two angular momenta with quantum numbers J_1, M_1 and J_2, M_2, the rules are

$$J = |J_1 - J_2|, |J_1 - J_2| + 1, \ldots, J_1 + J_2, \qquad M_J = M_1 + M_2$$

and again we cannot know J, M_1 and M_2 simultaneously.

Confusingly, when referring to a composite particle (eg a hadron or nucleus), the total angular momentum is often called its "spin" but given the quantum number J. Sometimes this usage even extends to elementary particles. For the electron and proton, s is more common though.

For the case of a spin-$\frac{1}{2}$ particle, the eigenvalues of $\widehat{S}_z$ are $\pm\frac{1}{2}\hbar$, and here we will just denote these states by $\uparrow$ and $\downarrow$ (α_z and β_z are also often used); hence

$$\begin{array}{cc} \widehat{\mathbf{S}}^2{\uparrow} = \dfrac{3}{4}\hbar^2{\uparrow} & \widehat{\mathbf{S}}^2{\downarrow} = \dfrac{3}{4}\hbar^2{\downarrow} \\ \widehat{S}_z{\uparrow} = \frac{1}{2}\hbar{\uparrow} & \widehat{S}_z{\downarrow} = -\frac{1}{2}\hbar{\downarrow} \end{array}$$

For two such particles there are four states $\uparrow\uparrow$, $\downarrow\downarrow$, $\uparrow\downarrow$ and $\downarrow\uparrow$. The first two states have $M_S = 1$ and -1 respectively, and we can show, using $\widehat{\mathbf{S}}^{\rm tot} = \widehat{\mathbf{S}}^{(1)} + \widehat{\mathbf{S}}^{(2)}$, that they are also eigenstates of $(\widehat{\mathbf{S}}^{\rm tot})^2$ with $S = 1$. However the second two, though they have $M_S = 0$, are not eigenstates of $(\widehat{\mathbf{S}}^{\rm tot})^2$. To make those, we need linear combinations, tabulated below:

	$S=1$	$S=0$
$M=1$	$\uparrow\uparrow$	
$M=0$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow+\downarrow\uparrow)$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)$
$M=-1$	$\downarrow\downarrow$	

The $S=1$ states are symmetric under exchange of particles; the $S=0$ states are antisymmetric. For a system of N spin-$\frac{1}{2}$ particles, S will be integer if N is even and half-integer if N is odd.

Bosons and Fermions

Particles with half-integer spin (electrons, baryons) are called fermions, those with integer spin, including $J=0$, (mesons, photons, Higgs) are called bosons. The "Pauli exclusion principle" applies to fermions, but it is a special case of the "spin-statistics theorem" which says that the overall quantum state of a system of identical fermions must be antisymmetric under exchange of any pair, while that of a system of identical bosons must be symmetric. There may be several components to the state (spatial wave function, spin state...).

Examples of the consequences of the spin-statistics theorem are:

- If two electrons in an atom are in the same orbital (thus their spatial wave function is symmetric under exchange of the two), they must be in an $S=0$ state.
- Thus the ground state of helium has $S=0$, but the excited states can have $S=0$ (parahelium) or $S=1$ (orthohelium).
- Two π^0 mesons must have even relative orbital angular momentum L (they are spinless, so this is the only contribution to their wave function).
- Two ρ^0 mesons (spin-1 particles) can have odd or even relative orbital angular momentum L, but their spin state must have the same symmetry as their spatial state. (In this case, $S=2$ and 0 are even, $S=1$ is odd.)

Note that in the last two, in the centre-of-momentum frame the spatial state only depends on the relative coordinate $\mathbf{r}$. So interchanging the particles is equivalent to $\mathbf{r}\to-\mathbf{r}$, ie the parity operation.

Spherical Harmonics

In spherical polar coordinates the orbital angular momentum operators are

$$\widehat{L}_x = \tfrac{1}{2}(\widehat{L}_+ + \widehat{L}_-) \quad \text{and} \quad \widehat{L}_y = \tfrac{1}{2i}(\widehat{L}_+ - \widehat{L}_-), \quad \text{where}$$

$$\widehat{L}_+ = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right), \qquad \widehat{L}_- = \widehat{L}_+^\dagger = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right);$$

$$\widehat{L}_z = -i\hbar\frac{\partial}{\partial\phi}, \qquad \widehat{\mathbf{L}}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right).$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\psi - \frac{1}{\hbar^2 r^2}\widehat{\mathbf{L}}^2\psi;$$

The spherical harmonics, $Y_l^m(\theta,\phi)$ are eigenfunctions of $\widehat{\mathbf{L}}^2$ and $\widehat{L}_z$; the first few are as follows

$$Y_0^0(\theta,\phi) = \sqrt{\frac{1}{4\pi}} \qquad\qquad Y_1^{\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\,\sin\theta\,e^{\pm i\phi}$$

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\,\cos\theta \qquad\qquad Y_2^{\pm2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\,e^{\pm2i\phi}$$

$$Y_2^{\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}\,\sin\theta\,\cos\theta\,e^{\pm i\phi} \qquad\qquad Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1)$$