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Universal behaviour of four-boson systems from a functional renormalisation group

Michael C Birse
The University of Manchester

Results from:

Jaramillo Avila and Birse, arXiv:1304.5454
Schmidt and Moroz, arXiv:0910.4586
Krippa, Walet and Birse, arXiv:0911.4608
Birse, Krippa and Walet, arXiv:1011.5852

Background

Ideas of effective field theory and renormalisation group

- well-developed for few-nucleon and few-atom systems
- rely on separation of scales
- Wilsonian RG used to derive power counting

$\rightarrow$ classify terms as perturbations around fixed point (or limit cycle)

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Many unsuccessful attempts to extend to dense matter

- but no separation of scales
- other EFT's for interacting Fermi systems exist (Landau Fermi liquid, Ginsburg-Landau theory)
- but parameters have no simple connection to underlying forces

EFTs based on contact interactions

- not well suited for standard many-body methods

$\rightarrow$ switch to lattice simulation or look for some more heuristic approach

- based on field theory
- can be matched onto EFT's for few-body systems (input from 2- and 3-body systems in vacuum)

Try functional renormalisation group ("exact" RG)

- based on Wilsonian RG approach to field theories
- successfully applied to various systems in areas from condensed-matter physics to quantum gravity [version due to Wetterich (1993)]

Outline

- Functional RG
- Spin-$\frac{1}{2}$ fermions
  - Dimer-dimer scattering
- Bosons
  - Efimov physics
  - 4-body systems

Functional RG

Version based on the effective action $\Gamma[\phi_c]$

- start from generating function $W[J]$ defined by

$$e^{iW[J]} = \int D\phi \, e^{i(S[\phi]+J\cdot\phi-\frac{1}{2}\phi\cdot R\cdot\phi)}$$

- $R(q,k)$: regulator function
  suppresses modes with momenta $q \lesssim k$ ("cutoff scale")
- only modes with $q \gtrsim k$ integrated out
- $W[J]$ becomes full generating function as $k \to 0$

Legendre transform $\to$ effective action

$$\Gamma[\phi_c] = W[J] - J\cdot\phi_c + \frac{1}{2}\phi_c\cdot R\cdot\phi_c \quad \text{where} \quad \phi_c = \frac{\delta W}{\delta J}$$

(generating function for 1-particle-irreducible diagrams)

Γ evolves with scale k according to

$$\partial_k \Gamma = -\frac{i}{2}\,\mathrm{Tr}\left[(\partial_k R)\left(\Gamma^{(2)} - R\right)^{-1}\right] \quad \text{where} \quad \Gamma^{(2)} = \frac{\delta^2 \Gamma}{\delta\phi_c \delta\phi_c}$$

$(\Gamma^{(2)} - R)^{-1}$: propagator of boson in background field ϕ_c
(one-loop structure but still exact)

Evolution interpolates between "bare" classical action at large scale K and full 1PI effective action as $k \to 0$ (thresholds etc ...)

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Functional differential equation

- hard/impossible solve in general

$\rightarrow$ work with tructated ansatz for Γ

- local action expanded in powers of derivatives
  (cf low-energy EFTs, but don't know *a priori* if we have a consistent power counting)

Derivative expansion may be good at starting scale K

- use power counting of EFT to determine relevant terms
  (or use this RG to find that power counting in scaling regime)
- but no guarantee that it remains good for $k \to 0$
  (can't be for scattering amplitudes at energies above threshold:
  cuts $\to$ nonanalytic behaviour)

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$\to$ need consistency checks:
stability against adding extra terms to ansatz
stability against changes in form of regulator

- use this to optimise choice of regulator [Litim, Pawlowski]

Two species of fermion

Fermion field: $\psi(x)$ (spin-$\frac{1}{2}$ atoms or neutrons)
Boson "dimer" field: $\phi(x)$ (strongly interacting pairs)
Local (nonrelativistic) ansatz for action in vacuum: 2-body sector

$$\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger;k]$$
$$= \int \mathrm{d}^4x \left[\psi(x)^\dagger\left(\mathrm{i}\,\partial_0 + \frac{\nabla^2}{2M}\right)\psi(x)\right.$$
$$+ Z_\phi(k)\,\phi(x)^\dagger\left(\mathrm{i}\,\partial_0 + \frac{\nabla^2}{4M}\right)\phi(x) - u_1(k)\,\phi(x)^\dagger\phi(x)$$
$$\left. - g\left(\frac{\mathrm{i}}{2}\phi(x)^\dagger\psi(x)^{\mathrm{T}}\sigma_2\psi(x) + \mathrm{H\,c}\right)\right]$$

g: AA$\to$D coupling
$u_1(k)$: dimer self-energy (u_1/g^2: only physical parameter)
$Z_\phi(k)$: dimer wave-function renormalisation

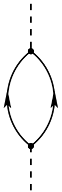
Evolution equation

$$\partial_k \Gamma = +\frac{i}{2}\,\mathrm{Tr}\left[(\partial_k \mathbf{R}_F)\left((\mathbf{\Gamma}^{(2)}-\mathbf{R})^{-1}\right)_{FF}\right] - \frac{i}{2}\,\mathrm{Tr}\left[(\partial_k \mathbf{R}_B)\left((\mathbf{\Gamma}^{(2)}-\mathbf{R})^{-1}\right)_{BB}\right]$$

$\mathbf{\Gamma}^{(2)}$: matrix of second derivatives of the action

(Gorkov-like form: ψ and $\psi^\dagger$ as independent variables $\rightarrow$ factors of $\frac{1}{2}$)

"Skeleton" diagram for driving terms in evolution of 2-body parameters



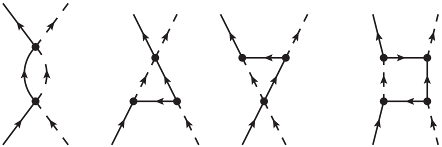
(need to insert $\partial_k \mathbf{R}_F$ on one internal line)

Expand in powers of energy $\rightarrow \partial_k u_1,\, \partial_k Z_\phi$

3-body sector: AD contact interaction

$$\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger;k] = \cdots - \lambda(k)\int d^4x\,\psi^\dagger(x)\phi^\dagger(x)\phi(x)\psi(x)$$

Evolution of λ driven by terms corresponding to skeletons



- AD contact interaction
- single-A exchange between dimers
  (cf Faddeev and STM equations)

4-body sector: DD→DD, DD→DAA, DAA→DAA terms

[Birse, Krippa and Walet (2010); cf Schmidt and Moroz (2009): bosons]

$$\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger;k] = \cdots - \int d^4x \left[\frac{1}{2} u_2(k)\left(\phi^\dagger\phi\right)^2 + \frac{1}{4} v(k)\left(\mathrm{i}\,\phi^{\dagger 2}\phi\,\psi^{\mathrm{T}}\sigma_2\psi + \mathrm{H\,c}\right) + \frac{1}{4} w(k)\,\phi^\dagger\phi\,\psi^\dagger\sigma_2\psi^{\dagger\mathrm{T}}\,\psi^{\mathrm{T}}\sigma_2\psi\right]$$

- dimer “breakup” terms allow 3-body physics to feed in properly (cf Faddeev-Yakubovski)

→ coupled evolution equations for u_2, v, w (27 distinct skeletons)

Regulators

- fermions: sharp cutoff

$$R_F(\boldsymbol{q},k)=\frac{k^2-q^2}{2M}\,\theta(k-q)$$

- pushes states with $q>k$ up to energy $k^2/2M$
- nonrelativistic version of "optimised" cutoff [Litim (2001)]
- fastest convergence at this level of truncation
- bosons

$$R_B(\boldsymbol{q},k)=Z_\phi(k)\,\frac{(c_B\,k)^2-q^2}{4M}\,\theta(c_B\,k-q)$$

- c_B: relative scale of boson cutoff
- optimised choice $c_B=1$ [cf Pawlowski (2007)]
  (no mismatch between fermion and boson cutoffs)

Also examined smooth cutoffs – more convenient in dense matter

Initial conditions

As $k \to \infty$ boson field purely auxiliary

- $Z_\phi(k) \to 0$
- $u_1(K)$ chosen so that in physical limit ($k \to 0$)

$$u_1(0) = -\frac{M g^2}{4\pi a_0} \qquad a_0\text{: AA scattering length}$$

- other couplings λ, u_2, v, w also vanish as $k \to \infty$

$\to$ either set $Z_\phi(K) = 0$ etc at large starting scale K
or match on to K^{-n} behaviour in scaling regime $K \gg 1/a_0$

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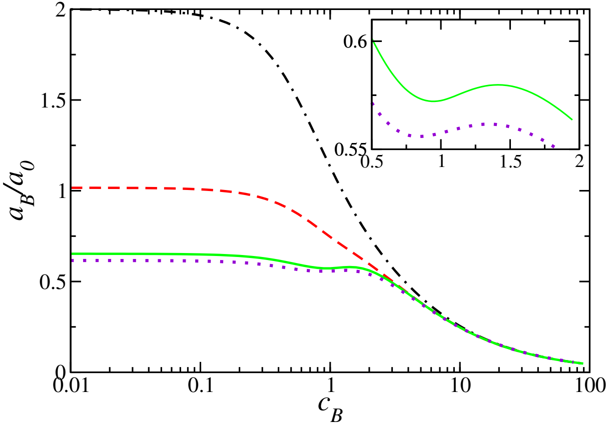
Expansion point for $a_0 > 0$: dimer binding energy $\mathcal{E}_D = -1/(M a_0^2)$

- external boson lines carry $P_0 = \mathcal{E}_D$
- external fermion lines carry $P_0 = \mathcal{E}_D/2$
  (below all thresholds)

Results: DD scattering length



- black: "minimal" action – only two-body and DD vertex u_2
- red adds three-body coupling λ
- green: full local four-body action, includes v, w
- purple: similar but using smooth cutoff

Comments

- results seem to converge as more terms are included
- converge to value only weakly dependent on cutoff
  (very little variation over range $0 \leq c_B \lesssim 2$)
- stationary very close to expected "optimum" $c_B = 1$
- incomplete actions $\rightarrow$ strong dependence on c_B around $c_B = 1$

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Final result

- $a_B/a_0 \simeq 0.58 \pm 0.02$
- agrees well with full few-body result $a_B/a_0 = 0.6$
  [Petrov, Salomon and Shlyapnikov (2004)]

Bosons

More interesting: Efimov effect in 3-body sector

Very similar action and evolution equations

- 3-body coupling λ periodic under scaling k by factor e^{π/s_0} where $s_0 = 0.92503$ [Schmidt and Moroz (2009)]
- agrees with Efimov $s_0 = 1.00624$ to $< 10\%$
- no sign of 4-body bound states at this truncation [numerical integration requires some care – poles in λ]

Introduce trimer field $\chi(x)$

- include energy dependence associated with 3-body bound states
- obtain equations with structure like Faddeev-Yakubovsky [coupled DD, AT channels]

Effective action

$$
\begin{aligned}
&\Gamma_k[\psi,\psi^*,\phi,\phi^*,\chi,\chi^*] \\
&= \int d^4x \Bigg[\psi^*\left(i\partial_0 + \frac{\nabla^2}{2m}\right)\psi + Z_d\,\phi^*\left(i\partial_0 + \frac{\nabla^2}{4m}\right)\phi + Z_t\,\chi^*\left(i\partial_0 + \frac{\nabla^2}{6m}\right)\chi \\
&\qquad - u_d\phi^*\phi - u_t\chi^*\chi - \frac{g}{2}\left(\phi^*\psi\psi + \psi^*\psi^*\phi\right) - h\left(\chi^*\phi\psi + \phi^*\psi^*\chi\right) \\
&\qquad - \lambda\,\phi^*\psi^*\phi\psi \\
&\qquad - \frac{u_{dd}}{2}\left(\phi^*\phi\right)^2 - \frac{v_d}{4}\left(\phi^*\phi^*\phi\psi\psi + \phi^*\psi^*\psi^*\phi\phi\right) - \frac{w}{4}\phi^*\psi^*\psi^*\phi\psi\psi \\
&\qquad - u_{tt}\,\chi^*\psi^*\chi\psi - \frac{u_{dt}}{2}\left(\phi^*\phi^*\chi\psi + \chi^*\psi^*\phi\phi\right) \\
&\qquad - \frac{v_t}{2}\left(\phi^*\psi^*\psi^*\chi\psi + \chi^*\psi^*\phi\psi\psi\right)\Bigg]
\end{aligned}
$$

AD interaction λ regenerated by evolution even if zero initially
[unlike AA scattering]

$\rightarrow$ introduce running trimer field [cf Gies and Wetterich (2002)]

$$\partial_k \chi = \zeta_1 \, \phi\psi + \zeta_2 \, \psi^\dagger \chi\psi + \zeta_3 \, \psi^\dagger \phi\phi + \zeta_4 \, \psi^\dagger \phi\psi\psi$$

where $\zeta_1 = -\partial_k \lambda / 2h$ to cancel running of λ

- other terms do same for four-atom couplings v_d, w and v_t
- additional piece in evolution equation

$$\partial_k \Gamma = -\frac{i}{2} \operatorname{Tr}\left[(\partial_k \mathbf{R}) \left((\mathbf{\Gamma}^{(2)} - \mathbf{R})^{-1} \right) \right] + \frac{\delta \Gamma}{\delta \chi} \cdot \partial_k \chi$$

- can keep λ, v_d, w and $v_t = 0$ for all k

3-body sector

Coupled equations for $u_t(k)$, $Z_t(k)$ and $h^2(k)$

Scaling limit $k \gg 1/|a_0|$

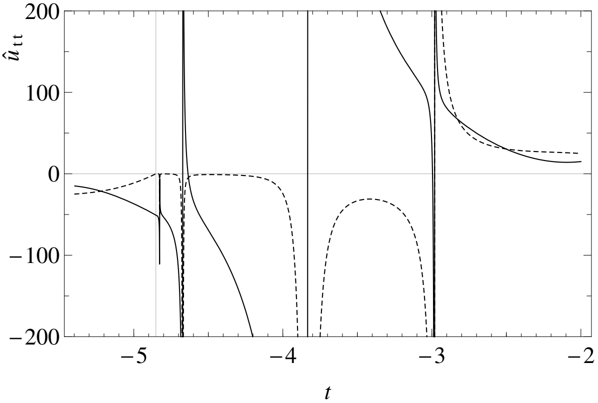
- couplings oscillate sinusoidally with $\ln k$
- poles in AD scattering amplitude h^2/u_t
  (values of k where 3-body bound states appear at zero energy)

→ tower of Efimov states with $s_0 = 0.92503$
  momentum scale factor $e^{\pi/s_0} = 29.2$ (exact: 22.7)

- tower cuts off when $k \sim 1/a_0$

4-body sector

3-body couplings $\rightarrow$ cyclic behaviour in $u_{dd}(k)$, $u_{dt}(k)$, and $v_t(k)$
One Efimov cycle of rescaled $\hat{u}_{tt}(k)$ as a function of $t = \ln(k/K)$

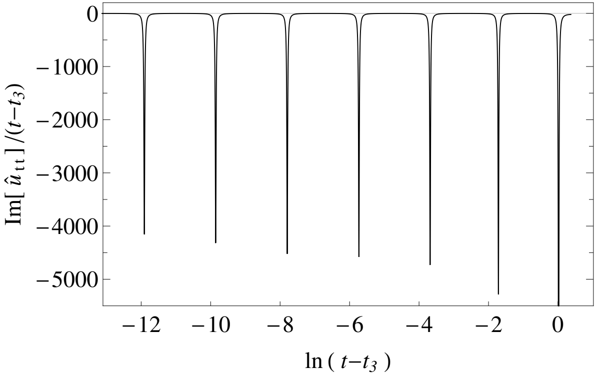


solid: real, dashed: imaginary
vertical grey line: AT threshold passes through zero energy

Comments

- imaginary part appears at AT threshold $t = t_3 \simeq -4.85$
- 4-body bound states below AT threshold $t \simeq -3.83, -4.67, \ldots$
  (decay to deeper trimer + free atom $\rightarrow$ finite widths)
- unphysical singularity from zero of $h^2(k)$ at $t \simeq -3.0$
  (end of region within cycle where $h^2(k)$, $Z_t(k)$ have opposite signs $\rightarrow$ trimer "ghost")

Infinite tower of 4-body bound states below AT threshold

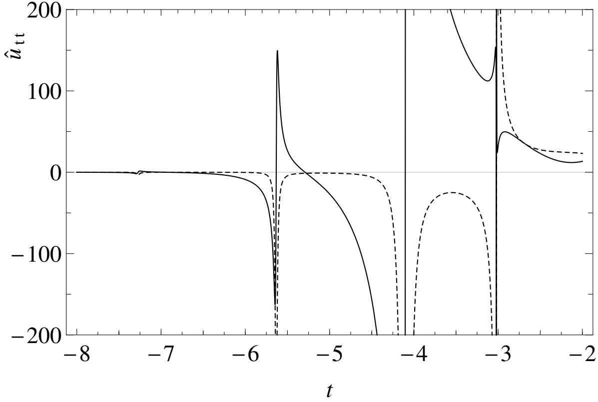


Double exponential pattern ∼ super-Efimov effect

[Nishida, Moroz and Son (2013)]

- but may not survive in physical limit $k \to 0$
- 4-body states may move relative to AT threshold, become virtual

Final cycle of $\hat{u}_{tt}(k)$ for finite $a_0 < 0$
tuned so that last three-body state appears at $k = 0$ ($t = -\infty$)



Three 4-body states, at $t = -4.1$, -5.6 and -7.1

(consistent with theorem of Amado and Greenwood)

AA scattering lenght corresponding to zero-energy 3-body state: a_3

(results just shown)

Further decrease in a: 4-body states cross zero energy at

$$a_4^{(0)}/a_3 \simeq 0.438, \quad a_4^{(1)}/a_3 \simeq 0.877, \quad a_4^{(2)}/a_3 \simeq 0.9967$$

Two lowest states: ratios within 5% of exact results

[von Stecher it et al (2009); Deltuva (2010)]

Third state extremely weakly bound

- if real: challenge to observe numerically and experimentally
- could be artefact of truncation
  (Efimov cycles too long: scale factor $\sim$ 30 instead of 23)

Scaling regime

Scales at which 3- and 4-body states appear

- double exponential form

$$k_4^{(n)} = k_3 \exp\left[\alpha\, e^{-\beta n}\right]$$

- Ratios of scales given by universal relation

$$k_4^{(n+1)}/k_4^{(n)} = \left(k_3/k_4^{(n)}\right)^{1-\exp(-\beta)}$$

Similar to universal scaling function found by Hadizadeh *et al* but

- different functional form
- no new 4-body scale parameter
  (α fixed, independent of the initial conditions)

Summary

Applications of functional RG to 3- and 4-body systems

- local effective action, “optimised” cutoff
- keeping all local terms in 4-body sector

Fermions

- results for dimer-dimer scattering length
  stable against variation of cutof, agree with direct calculations

Bosons

- dynamical trimer field to match structure of Faddeev-Yakubovsky equations
- imaginary parts of 4-body couplings from each AT threshold
- infinite tower of 4-body bound states below each AT threshold
- double exponential (super-Efimov) pattern
- finite 2-body scattering length: three states in last cycle

Super-Efimov effect

Relies on being close to fixed point with complex scaling
Example for fewer-body coupling g^2 at nontrivial fixed point

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a\,g^4 + b\,g^2 v + c\,v^2$$

with $b^2 - 4ac < 0 \to$ imaginary scaling dimension
Now consider g^2 marginal: $g^2 = g_0^2/t$ with $t = \ln(k/k_0)$
and define $\hat{v} = t\,v$

$$t\,\frac{\mathrm{d}\hat{v}}{\mathrm{d}t} = a\,g_0^4 + (1 + b\,g_0^2)\hat{v} + c\,\hat{v}^2$$

$\to$ cyclic behaviour in $\ln t = \ln(\ln(k/k_0))$ if

$$\left(\frac{1}{g_0^2} + b\right)^2 - 4ac < 0$$

4 bosons – close to AAD Efimov cycle [Deltuva (2012)]