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Functional RG for few-body systems

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Work in progress, based on: arXiv:0801.2317

Background

Ideas of effective field theory and the renormalisation group are now well-developed for few-body systems

- rely on separation of scales
- RG can be used to derive power counting

$\longrightarrow$ classify terms as perturbations around a fixed point (Wilsonian approach, sharp cut-offs)

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$\longrightarrow$ classify terms as perturbations around a fixed point (Wilsonian approach, sharp cut-offs)

Two-body scattering by short-range forces $\longrightarrow$ two fixed points

- trivial: no scattering
- nontrivial: zero-energy bound state (scale free) [Birse, McGovern and Richardson, hep-ph/9807302]

$\longrightarrow$ can describe nuclear forces at low energies or atomic systems with Feshbach resonance tuned to threshold

Perturbations around trivial fixed point

- RG eigenvalues $\nu = d + 1$
  d: naive dimension of operator (net power of low-energy scales)
- all irrelevant (vanish like Λ^ν as cut-off $\Lambda \to 0$)
- "Weinberg" power counting (like chiral pertubation theory)

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Perturbations around nontrivial fixed point

- energy-dependent: $\nu = d - 1$

$\longrightarrow$ fixed point is unstable (one relevant perturbation $\propto \Lambda^{-1}$)

- correspond to terms in effective-range expansion [Bethe, 1949]

Three particles with two-body bound states near zero energy

- noninteger RG eigenvalues for three-body forces
  in general less relevant than in naive dimensional analysis
  [Griesshammer, nucl-th/0502039; Birse, nucl-th/0509031]

$\longrightarrow$ low-energy three-body scattering determined by two-body scattering length

spin-3/2 neutron-deuteron scattering
[Bedaque and van Kolck, nucl-th/9710073]
three identical spin-1/2 atoms
[Diehl, Krahl and Scherer, arXiv:0712:2846]

More interesting: more than two “species” of fermion, or three bosons (spin-1/2 neutron-deuteron scattering, triton)

- RG flow tends to limit cycle [Bedaque, Hammer and van Kolck, arXiv:nucl-th/9809025; Głazek and Wilson, cond-mat/0303297; Barford and Birse, nucl-th/0406008]

$\longrightarrow$ Efimov effect (infinite tower of bound states with constant ratio between energies: $\sim$ scale-free) [Efimov, 1971]

- leading three-body force is marginal (fixes starting point on cycle or energy of one bound state)
- two-body data relates three-body scattering length and bound state energy (Phillips line)

$\longrightarrow$ one piece of three-body information required to fix low-energy observables

Many unsuccessful attempts to extend to dense fermionic matter (nuclear matter or cold trapped atoms)

- problem: no separation of scales
- only consistent EFT so far: weakly repulsive Fermi gas (reproduces old results of Bishop and others) [Hammer and Furnstahl, nucl-th/0004043]

Other EFT's for interacting Fermi systems exist:

- Landau Fermi liquid, Ginsburg-Landau theory
- but parameters have no simple connection to underlying forces (like ChPT and QCD)

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Look for some more heuristic approach

- based on field theory
- can be matched onto EFTs for few-body systems
- input from two-body (and 3- or 4-body) systems in vacuum

Promising approach: functional ("exact") renormalisation group

- successfully applied to various systems in particle and condensed-matter physics
  [version due to Wetterich, Phys Lett **B301** (1993) 90]
- interpolates between bare "classical" action
  and full effective action

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Recent applications

- fermionic matter: Birse *et al*, hep-ph/0406249;
  Diehl *et al*, cond-mat/0701198, cond-mat/0703366;
  Krippa, nucl-th/0605071, arXiv:0706.4000
- two-body scattering: Harada *et al*, nucl-th/0702074
- three-body scattering: Diehl, Krahl and Scherer, arXiv:0712.2846

Outline

- Effective action for nonrelativistic fermions
- Functional RG for two-body scattering
- Possible problems
- Four-body systems
- Summary

Effective action

Nonrelativistic fermions, field $\psi(x)$

- spin-1/2 (two "species")
- strong S-wave attraction $\rightarrow$ represented by pair boson field $\phi(x)$

Ansatz for action in vacuum ($\widetilde{\psi}(q)$, $\widetilde{\psi}(q)$: Fourier transforms of fields)

$$
\begin{aligned}
&\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger;k] \\
&= \int \mathrm{d}^4 q \left[\widetilde{\phi}(q)^\dagger\, \Pi(q_0,\boldsymbol{q};k)\,\widetilde{\phi}(q) + \widetilde{\psi}(q)^\dagger \left(q_0 - \frac{\boldsymbol{q}^2}{2M}\right)\widetilde{\psi}(q)\right] \\
&\quad -g\,\frac{1}{(2\pi)^2}\int \mathrm{d}^4 q_1\,\mathrm{d}^4 q_2 \left(\frac{\mathrm{i}}{2}\,\widetilde{\phi}(q_1+q_2)^\dagger\,\widetilde{\psi}(q_2)^{\mathrm{T}}\sigma_2\widetilde{\psi}(q_1)\right. \\
&\qquad\qquad \left. -\frac{\mathrm{i}}{2}\,\widetilde{\psi}(q_1)^\dagger\sigma_2\widetilde{\psi}(q_2)^{\dagger\mathrm{T}}\widetilde{\phi}(q_1+q_2)\right)
\end{aligned}
$$

Same action as used in studies of fermionic matter, except

- boson self-energy $\Pi(q_0, \boldsymbol{q}; k)$ not truncated in powers of energy, momentum
- no renormalisation of fermion propagator or coupling constant g in vacuum (only in superfluid—boson condensate)

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No two-body interaction between fermions

- expressed in terms of boson field
- auxiliary field, not dynamical at starting scale K

$$\Pi(q_0, \boldsymbol{q}; K) = -u_1(K)$$

- becomes dynamical as fluctuations are included

$$\Pi(q_0, \boldsymbol{q}; k) = -u_1(k) + Z_\phi(k) q_0 - Z_m(k) \frac{\boldsymbol{q}^2}{4M} + \cdots$$

Regulator $\mathbf{R}(k)$

- suppresses contributions of modes with low momenta, $|\boldsymbol{q}| \lesssim k$ ("cut-off")

$\longrightarrow$ action evolves with regulator scale k

becomes full effective action as $k \to 0$

Legendre-transformed action Γ (generator for 1PI diagrams) evolves according to "one-loop" RG equation

$$\begin{aligned} \partial_k \Gamma &= +\frac{\mathrm{i}}{2} \operatorname{Tr}\left[(\partial_k \mathbf{R}_F) \left((\Gamma^{(2)} - \mathbf{R})^{-1} \right)_{FF} \right] \\ &\quad -\frac{\mathrm{i}}{2} \operatorname{Tr}\left[(\partial_k \mathbf{R}_B) \left((\Gamma^{(2)} - \mathbf{R})^{-1} \right)_{BB} \right] \end{aligned}$$

$\Gamma^{(2)}$: matrix of second derivatives of the action

Convenient choice of regulator for fermions

- add term to single-particle energies

$$R_F(\boldsymbol{q},k)=\frac{k^2-q^2}{2M}\,\theta(k-q)$$

- nonrelativistic (three-momentum) version of Litim's "optimised" cut-off [Litim, hep-th/0103195]
- sharp cut-off: no effect on states with $q>k$
- simple energies for $q<k$: constant

Boson self-energy and two-body scattering

Evolution given by

$$\partial_k \Pi(P_0, \boldsymbol{P}; k) = \frac{\delta^2}{\delta\widetilde{\phi}(P)\,\delta\widetilde{\phi}(P)^\dagger}\,\partial_k \Gamma\bigg|_{\tilde{\phi}=0}$$

Vacuum: fermion loops only, energy integral straightforward

$\longrightarrow$ driving term: derivative with respect to k

$$\frac{\delta^2}{\delta\widetilde{\phi}(P)\,\delta\widetilde{\phi}(P)^\dagger}\,\partial_k \Gamma\bigg|_{\tilde{\phi}=0}$$
$$= g^2 \partial_k \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\,\frac{1}{E_{FR}(\boldsymbol{q}-\boldsymbol{P}/2, k) + E_{FR}(\boldsymbol{q}+\boldsymbol{P}/2, k) - P_0 - i\varepsilon}$$

regulated single-particle energy: $E_{FR}(\boldsymbol{q}, k) = \boldsymbol{q}^2/2M + R_F(\boldsymbol{q}, k)$

Easy to integrate with respect to k

(solving two-body Schrödinger equation, piecewise)

$$\Pi(P_0,P;k) = \Pi(P_0,P;0) - \frac{g^2 M}{4\pi^2}\left\{ i\pi\sqrt{MP_0 - P^2/4} \right.$$
$$- \sqrt{MP_0 - P^2/4}\,\ln\left(\frac{k + P/2 + \sqrt{MP_0 - P^2/4}}{k + P/2 - \sqrt{MP_0 - P^2/4}}\right)$$
$$+ \frac{1}{k^2 - MP_0}\left[\frac{7}{3}k^3 - 4kMP_0 - 3k^2P + \frac{5}{2}MP_0P - \frac{P^3}{24}\right]$$
$$+ 4\sqrt{k^2 - 2MP_0}\left[\arctan\left(\frac{k+P}{\sqrt{k^2 - 2MP_0}}\right)\right.$$
$$\left. - \arctan\left(\frac{k}{\sqrt{k^2 - 2MP_0}}\right)\right]$$
$$\left. - \frac{k^2 - P^2 - MP_0}{P}\ln\left[\frac{k^2 + kP + P^2/2 - MP_0}{k^2 - MP_0}\right]\right\}$$

Fermion-fermion scattering amplitude $T(p)$

- related to physical boson self-energy ($k \longrightarrow 0$)

$$T(p) = \frac{g^2}{\Pi(P_0, P, 0)}$$

- on-shell relative momentum $p = \sqrt{MP_0 - P^2/4}$
- effective-range expansion

$$\frac{1}{T(p)} = -\frac{M}{4\pi}\left(-ip - \frac{1}{a} + \frac{1}{2}r_e p^2 + \cdots\right)$$

- shows that RG generates correct threshold cut ($\propto ip$)

$\longrightarrow$ $\Pi(P_0, P; k)$ real for large k
expandable in powers of energy and momentum

Issues:

- Galilean invariance violated at order Q^3 and higher (regulator not invariant)
- unphysical nonanalytic term at order Q^3 consequence of nonlocalities introduced by sharp cut-off [Morris, hep-th/9308265]

$\longrightarrow$ will need to be addressed in matter calculations beyond current level of truncation

Input at starting scale $K \longrightarrow$ effective two-body potential

$$\frac{1}{V(p,P;K)} = \frac{1}{g^2}\Pi\left((p^2+P^2/4)/M,P;K\right)$$
$$= \frac{M}{4\pi^2}\left\{-\frac{4}{3}K+\frac{\pi}{a}+\left(\frac{8}{3K}-\frac{\pi}{2}r_e\right)p^2-\frac{1}{24K^2}P^3+\cdots\right\}$$

To study scaling behaviour

- express all dimensioned quantities in units of K

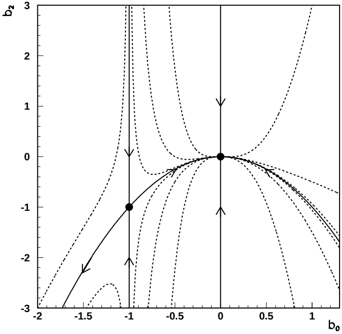
$\rightarrow$ define $\hat{p} = p/K$, $\hat{P} = P/K$
and rescaled potential $\hat{V} = (MK/2\pi^2)V$

$$\frac{1}{\hat{V}(\hat{p},\hat{P};K)} = -\frac{2}{3} + \frac{4}{3}\hat{p}^2 - \frac{1}{48}\hat{P}^3 + \cdots + K^{-1}\frac{\pi}{2a} - K\frac{\pi}{4}r_e\hat{p}^2 + \cdots$$

- nontrivial fixed point as found with Wilsonian RG
  [Birse *et al*, hep-ph/9807302]
- perturbations from effective-range expansion
- leading one is unstable (eigenvalue $\nu = -1$)

RG flow

Wilsonian version $\hat{V} = b_0 + b_2 \hat{p}^2 + \cdots$



- trivial and nontrivial fixed points
- critical line $1/a = \infty$ (zero-energy bound state)
- finite a: flow eventually heads to trivial point

Effective-range expansion

- encoded in coefficients of energy-dependent terms
- example: effective range r_e in wave-function renormalisation

$$Z_\phi(K) = \frac{\partial}{\partial P_0}\Pi(P_0, P; K)\Big|_{P_0=P=0} = \frac{g^2M^2}{4\pi^2}\left(\frac{8}{3K} - \frac{\pi}{2}r_e\right)$$

(negative if r_e positive and starting K too large)

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Off-shell behaviour

- controlled by momentum-dependent terms
- (Wilsonian) RG eigenfunctions: “equation of motion” form
- near nontrivial fixed point:
  less relevant than corresponding energy-dependent ones
- evidence for new highly unstable fixed points
  [Harada, Kubo and Ninomiya, nucl-th/0702074]

More than two particles

Three-body systems Diehl *et al*, arXiv0712.2846

Four-body systems

- add boson-boson scattering term to action

$$-\frac{1}{2}u_2\left(\phi^\dagger\phi\right)^2$$

  (as in matter calculations)
- describes $2+2$ part of Faddeev-Yakubowsky equations
  ($3+1$ still missing)
- scaling analysis at two-body fixed point
  $\longrightarrow$ stable nontrivial fixed point
  [Diehl *et al*, cond-mat/0701198]

Finite two-body scattering length a

- RG flow never reaches these fixed points

  $\longrightarrow$ either weakly interacting fermions

  (energies near breakup threshold)

  $\longrightarrow$ or tightly bound but weakly interacting bosons

  (energies near two-body bound-state)

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To reach bosonic EFT

- need to integrate through region where $\Pi(P_0, P; k)$ develops complicated nonanalytic dependence on P_0, k
- no numerical implementation yet
- one suggestion: integrate out fermions first
  then match onto purely bosonic theory [Diehl *et al*]
  but at what scale?

Summary

Functional RG equation (Legendre-transformed version)

- convenient tool for studying two-body systems
- can be solved exactly for two-body scattering
  (boson self-energy)

$\longrightarrow$ reproduces fixed-points and power counting found using Wilsonian RG

$\longrightarrow$ extends results to nonzero total momentum

- highlights issues that will need to be addressed in improved applications to dense fermionic matter
  - violations of Galilean invariance
  - nonanalytic terms generated by sharp cut-offs
  - problems with taking starting scale too high
- first applications to three-, four-body systems