PHYS30201 Advanced Quantum Mechanics: Examples 4

26. By explicit calculation, find the expression for the fine-structure shift in the energy of the $4\,^2p_{3/2}$ state of $^3\mathrm{Li}^{++}$.

27. Calculate the weak-field Zeeman shifts for all the $n = 1$ and $n = 2$ states of hydrogen and draw a level diagram. Mark all the transitions consistent with the $\Delta l = 1$, $\Delta m_j = -1, 0, 1$ rule, and list the frequencies.
Calculate the strong-field shifts and draw the levels for this case (ignoring fine-structure).

28. Find the magnitude of the electric field which would be required to induce a splitting of 1 meV between the $n = 2$ states of hydrogen. How large would the shift in the ground state be for that field?

29. (Challenge question) The following trick allows an exact calculation of the Stark shift for the ground state of hydrogen.
Starting with the more general formalism, suppose we find an operator $\hat{\Omega}$ such that $\widehat{H}^{(1)}|0^{(0)}\rangle = [\hat{\Omega}, \widehat{H}^{(0)}]|0^{(0)}\rangle$. Show that we can write

$$E_0^{(2)} = \sum_{m\neq 0} \frac{\langle 0^{(0)}|\widehat{H}^{(1)}|m^{(0)}\rangle\langle m^{(0)}|[\hat{\Omega}, \widehat{H}^{(0)}]|0^{(0)}\rangle}{E_0^{(0)} - E_m^{(0)}} = \langle 0^{(0)}|\widehat{H}^{(1)}\hat{\Omega}|0^{(0)}\rangle - E_0^{(1)}\langle 0^{(0)}|\hat{\Omega}|0^{(0)}\rangle.$$

(Hint - use $\sum_{m\neq 0}|m^{(0)}\rangle\langle m^{(0)}| = \hat{I} - |0^{(0)}\rangle\langle 0^{(0)}|$.)
Now show that if $\widehat{H}^{(1)} = -e\mathcal{E}z$, the operator $\hat{\Omega} = (ma_0 e\mathcal{E}/\hbar^2)(\frac{1}{2}r^2 + a_0 r)\cos\theta$ works (don't try to construct it, just verify the relation it needs to satisfy).
Hence calculate $E_0^{(2)}$.

30. There are eight $n = 2$ states in hydrogen. List them both in the $\{|ljm_j\rangle\}$ and $\{|lm_l\rangle \otimes |sm_s\rangle\}$ bases (we suppress the $n = 2$ label). Four in each set (those with $l = 0$ and with $m_j = \pm 3/2$) are in one-to-one correspondence; as we saw in questions 7 and 24 the other four mix pair-wise as follows:

$$\begin{aligned}
|1\tfrac{1}{2}\tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}}|11\rangle \otimes |\tfrac{1}{2}{-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{1}{3}}|10\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle \\
|1\tfrac{3}{2}\tfrac{1}{2}\rangle &= \sqrt{\tfrac{1}{3}}|11\rangle \otimes |\tfrac{1}{2}{-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{2}{3}}|10\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle \\
|1\tfrac{1}{2} - \tfrac{1}{2}\rangle &= \sqrt{\tfrac{1}{3}}|10\rangle \otimes |\tfrac{1}{2}{-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{2}{3}}|1{-1}\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle \\
|1\tfrac{3}{2} - \tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}}|10\rangle \otimes |\tfrac{1}{2}{-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{1}{3}}|1{-1}\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle
\end{aligned}$$

In the absence of an external field the four $j = 1/2$ states $|0\tfrac{1}{2} \pm \tfrac{1}{2}\rangle$ and $|1\tfrac{1}{2} \pm \tfrac{1}{2}\rangle$ are degenerate, since the fine-structure splits them from the $j = 3/2$ states. Use degenerate perturbation theory in this space to show that for a very weak electric field the Stark effect shifts the energy levels by $\mp\sqrt{3}a_0 e\mathcal{E}$).

31. A tritium atom beta-decays to $^3He^+$. What is the probability that the electron is in the ground state of $^3He^+$ after the decay?

32. Consider a system consisting of a spin-$\frac{1}{2}$ particle (magnetic moment μ) in a constant magnetic field B_0 along the z-axis, to which another rotating magnetic field in the xy-plane is applied. If the rotating field is $B_a(\cos(\omega t), -\sin(\omega t), 0)$, show that the perturbation can be written

$$\hat{H}^{(1)} = -\mu B_a \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}$$

in the basis of the eigenstates of $\widehat{H}^{(0)}$ with energies $\mp\mu B_0$.

Hence show that the equations of time-dependent perturbation theory, without any approximations, are

$$\dot{d}_1(t) = i\gamma e^{i(\omega-\omega_0)t} d_2(t) \qquad \dot{d}_2(t) = i\gamma e^{-i(\omega-\omega_0)t} d_1(t)$$

where $\omega_0 = 2\mu B_0/\hbar$ and $\gamma = \mu B_a/\hbar$. By differentiating the first of these w.r.t. time and substituting to eliminate d_2 and $\dot{d}_2$, obtain an uncoupled differential equation for $d_1(t)$ and solve it subject to the boundary conditions $d_1(0) = 0$, $d_2(0) = 1$, to get

$$P_1(t) \equiv |d_1(t)|^2 = \frac{\gamma^2}{\frac{1}{4}(\omega-\omega_0)^2 + \gamma^2} \sin^2\left(t\sqrt{\tfrac{1}{4}(\omega-\omega_0)^2 + \gamma^2}\right).$$

Sketch the maximum value of P_1 as a function of ω and discuss the result.

33. Consider a one-dimensional harmonic oscillator with a perturbation $\widehat{H}^{(1)} = \lambda\hat{x}e^{-t^2/\tau^2}$. Use first-order perturbation theory to show that if the system started off in the ground state at $t = -\infty$, the probability of a transition to the first excited state by $t = \infty$ is

$$P_{0\to 1} = \frac{\lambda^2\tau^2\pi}{2m\hbar\omega} e^{-\tau^2\omega^2/2}$$

Comment on the $\tau \to 0$ and $\tau \to \infty$ limits of this expression.