PHYS30201 Advanced Quantum Mechanics: Examples 3

15. Use the WKB approximation to find values for the energy levels of the potential $\beta\,|x|$.

16. Fill in the details to obtain the result given in lectures for the energy levels of the harmonic oscillator in the WKB approximation. (You will need to show that $\int \sqrt{c^2-x^2}\,\mathrm{d}x = (1/2)\big(x\sqrt{c^2-x^2}+c^2\arcsin(x/c)\big)$; use the substitution $x = c\sin\theta$.)

    Show that the Schrödinger equation for a particle in a spherical harmonic well $V(r) = \frac{1}{2}m\omega^2 r^2$ with $l = 0$ is equivalent to a one-dimensional Schrödinger equation for $u(r) = r\psi(r)$, subject to the boundary condition that $u(0) = 0$. Hence show (with negligible extra work) that the energy levels of this system in the WKB aproximation are $E = (2n+\frac{3}{2})\,\hbar\omega$. (This is exact in fact – make sure you can explain it in terms of your prior knowledge of the harmonic oscillator.)

17. Use the WKB approximation to show that the fusion probability for two protons in a head-on collision with center-of-mass energy E is, roughly, $\exp[-(r_c/R_G)^{1/2}]$, where R_G is the energy-independent Gamow radius, $R_G = \hbar/(\pi^2 m_p c\alpha)$, and $r_c(E)$ is the classical closest approach distance. (Assume r_c is much greater than the proton radius.) Comment on the relevance of the numbers for fusion in the sun.

18. Use the WKB approximation to find the dominant energy dependence for tunnelling through a quadratic barrier $V(x) = -\frac{1}{2}m\omega^2 x^2$ for a particle with energy much less than zero. (A realistic barrier would flatten out for $|x| > L$ for some value of L, so there is an overall energy offset of $-\frac{1}{2}m\omega^2 L^2$. This is irrelevant to the algebra of the problem.)

19. A particle mass m moves in the 1-D potential given by $V(x) = \infty$ for $x < 0$ and $x > a$, and $V(x) = -\lambda\sin(\pi x/a)$ for $0 < x < a$. Treating this as a perturbation on an infinite square well, calculate the first order shift in the ground state energy.

20. A two state system, degenerate under $\widehat{H}^{(0)}$, is subject to two separate perturbations $\widehat{H}^{(1)}_a$ and $\widehat{H}^{(1)}_b$. In an appropriate basis, the corresponding matrices are

    $$\widehat{H}^{(0)} = \begin{pmatrix} E^{(0)} & 0 \\ 0 & E^{(0)} \end{pmatrix}, \qquad \widehat{H}^{(1)}_a = \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}, \qquad \widehat{H}^{(1)}_b = \begin{pmatrix} b & -b \\ -b & b \end{pmatrix}.$$

    For $a \gg b$, treat $\widehat{H}^{(1)}_a$ as part of $\widehat{H}^{(0)}$ and calculate the first and second order energy shifts due to $\widehat{H}^{(1)}_b$.
    For $a \ll b$, treat $\widehat{H}^{(1)}_b$ as part of $\widehat{H}^{(0)}$. Diagonalise the matrix in that basis, then calculate the first and second order energy shifts due to $\widehat{H}^{(1)}_a$.

    Find the eigenvalues of $\widehat{H}^{(0)} + \widehat{H}^{(1)}_a + \widehat{H}^{(1)}_b$ exactly. Show that the previous results are obtained in the appropriate limits, and explain what goes wrong with them as b and a approach the same magnitude.

21. A three-state system has unperturbed Hamiltonian $\widehat{H}^{(0)}$ and is subject a perturbation $\widehat{H}^{(1)}$. In the basis of the eigenstates of $\widehat{H}^{(0)}$, the corresponding matrices are given below.

$$\widehat{H}^{(0)} = \begin{pmatrix} E_1^{(0)} & 0 & 0 \\ 0 & E_2^{(0)} & 0 \\ 0 & 0 & E_3^{(0)} \end{pmatrix}, \qquad \widehat{H}^{(1)} = \begin{pmatrix} 0 & b & a \\ b & 0 & a \\ a & a & 0 \end{pmatrix}.$$

Calculate the first order energy shifts and eigenstate shifts of all three states:
i) when $E_1^{(0)}$, $E_2^{(0)}$ and $E_3^{(0)}$ are all different;
ii) when $E_1^{(0)} = E_2^{(0)}$.
Compare with the exact results in the second case.

22. Consider a 1-D harmonic oscillator with perturbation $\widehat{H}^{(1)} = \lambda x$. Show that there is no first-order energy shift for any state. Calculate the first-order shift in the wavefunction, and the second order energy shift, for all states. Check your results against the exact results from the first examples sheet.

Now repeat for $\widehat{H}^{(1)} = \lambda x^3$. (Aside: do you believe that the results will be reliable regardless of n? Hint - sketch the potential).

23. The unperturbed states of the symmetric 2-D harmonic oscillator can be denoted $|n_x n_y\rangle$ with energy $E^{(0)} = (n_x + n_y + 1)\hbar\omega$, and states with the same $n_x + n_y$ are degenerate. Now consider a perturbation $\widehat{H}^{(1)} = \lambda xy$. Show that the states with $E^{(0)} = 2\hbar\omega$ will be mixed by this perturbation, and that in the subspace of these states the perturbation can be written

$$\frac{\hbar\lambda}{2m\omega}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and hence find the perturbed energies to first order in λ.

24. A system consists of a spin-1 and a spin-1/2 component. The unperturbed ground state of the system has zero energy and is insensitive to the spin-components; it is thus 6-fold degenerate. We denote the unperturbed kets by the m_s values of the individual spins, $\{|m_s^{(1)}, m_s^{(2)}\rangle\}$. A perturbation is now applied: $H^{(1)} = \lambda\widehat{\mathbf{S}}^{(1)} \cdot \widehat{\mathbf{S}}^{(2)}$. Use non-degenerate perturbation theory to find the first-order energy shifts of the states $|1, \frac{1}{2}\rangle$ and $|-1, -\frac{1}{2}\rangle$. Show that the perturbation mixes the states $|1, -\frac{1}{2}\rangle$ and $|0, \frac{1}{2}\rangle$, and find the 2×2 matrix of $H^{(1)}$ in this subspace. Find its eigenstates and eigenvalues, and comment on the results.

Hint: You will find the work you did in question 5 very useful here. Note the change in notation though: there we wrote the basis states as $\{|1, m_s^{(1)}\rangle \otimes |\frac{1}{2}, m_s^{(2)}\rangle\}$ and the perturbing operator as $\lambda \sum_i \widehat{S}_i^{(1)} \otimes \widehat{S}_i^{(2)}$. We do not normally use the explicit vector product notation, but it is always implied in cases like this one.

25. A sytem has orbital angular momentum l and spin s. We consider states which are eigenfunctions of $\widehat{\mathbf{J}}^2$ and $\widehat{J}_z$ rather than $\widehat{S}_z$ and $\widehat{L}_z$, denoted $|lsjm_j\rangle$. In lectures, we will use without proof the result that $\langle lsjm_j|\hat{S}_z|lsjm_j\rangle$ must be proportional to $\langle lsjm_j|\hat{J}_z|lsjm_j\rangle$. Here we demonstrate that this is true for spin-$\frac{1}{2}$ systems. Using

$$|l\tfrac{1}{2}jm_j\rangle = \pm\sqrt{\frac{l + \frac{1}{2} \pm m_j}{2l+1}}|l, m_j - \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle + \sqrt{\frac{l + \frac{1}{2} \mp m_j}{2l+1}}|l, m_j + \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2} - \tfrac{1}{2}\rangle$$

for $j = l \pm \frac{1}{2}$, find $\langle l\frac{1}{2}jm_j|\hat{S}_z|l\frac{1}{2}jm_j\rangle$ and show that it is equal to $\hbar m_j\langle\widehat{\mathbf{S}} \cdot \widehat{\mathbf{J}}\rangle/\langle\widehat{\mathbf{J}}^2\rangle$. (Hint: deal with the cases $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ separately.)