PHYS30201 Advanced Quantum Mechanics: Examples 1 - Revision

Most of these questions are revision; you will find sections A.1-A.4 of the web notes useful for these.

1. (a) Prove that the eigenvalues $\{\omega_i\}$ of a Hermitian operator $\widehat{\Omega}$ are real and the (normalised, non-degenerate) eigenvectors $\{|\omega_i\rangle\}$ are orthogonal.
   (b) If $|\alpha\rangle = \sum_i a_i|\omega_i\rangle$ and $|\beta\rangle = \sum_i b_i|\omega_i\rangle$, show that $\langle\beta|\alpha\rangle = \sum_i b_i^* a_i = (\langle\alpha|\beta\rangle)^*$.
   (c) Prove also that $\sum_i |\omega_i\rangle\langle\omega_i| = \widehat{I}$. (Hint - show that acting on an arbitrary ket $|\alpha\rangle$ gives $|\alpha\rangle$ again.)
   (d) Another Hermitian operator $\widehat{\Theta}$ has matrix elements $\langle\omega_i|\widehat{\Theta}|\omega_j\rangle = \theta_{ij}$. Show that $\langle\beta|\widehat{\Theta}\alpha\rangle = \sum_{ij}\theta_{ij}b_i^* a_j = \langle\widehat{\Theta}\beta|\alpha\rangle$, and also that $\sum_{ij}|\omega_i\rangle\theta_{ij}\langle\omega_j| \equiv \widehat{\Theta}$.

2. A particle is in a state $|\psi\rangle$ when the observable corresponding to $\widehat{\Omega}$ is measured. Show that the following two statements are equivalent (i.e. assume one and show that the other is a consequence of it):
   (I) the probability of getting a result ω_i is $|\langle\omega_i|\psi\rangle|^2$;
   (II) the expectation value (ensemble average) is $\langle\psi|\widehat{\Omega}|\psi\rangle$.

3. (a) Verify that $\langle\alpha|\beta\rangle = \int \alpha(x')^*\beta(x')\mathrm{d}x'$. (Hint - use the appropriate identity operator.)
   (b) Verify that $\langle x|\hat{p}|x'\rangle = i\hbar\frac{\mathrm{d}}{\mathrm{d}x'}\delta(x-x')$ implies $\langle x|\hat{p}|\alpha\rangle = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\alpha(x)$. (Hint - as above!)
   (c) Verify using the previous result that $\langle x|[\hat{x},\hat{p}]|\alpha\rangle = i\hbar\alpha(x)$ (ie not just by replacing the commutator by $i\hbar$).
   (d) Verify that if $\tilde{\alpha}(p)$ is the Fourier transform of $\alpha(x)$, $p\tilde{\alpha}(p)$ is the Fourier transform of $-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\alpha(x)$. (Use the convention that has $-ikx$ in the exponent for the transform, and ikx for the inverse.)
   (e) If $\phi(x) = (\pi x_0^2)^{-1/4}\exp(-x^2/(2x_0^2))$, find $\tilde{\phi}(p)$ and check the normalisation.

4. A hydrogen atom is prepared in an initial state described by the wave function

$$\psi_I(\mathbf{r}) = \frac{1}{\sqrt{96\pi a_0^5}}\, r\exp\left(-\frac{r}{2a_0}\right).$$

   A measurement is made of the energy. What is the probability of obtaining the ground state energy of -13.6 eV? What other values do you think might be obtained? What would the probability be if instead the initial state were $\sqrt{3}\cos(\theta)\psi_I(\mathbf{r})$? (Hint: You will need to know the ground-state wave function of hydrogen to answer this. The first answer works out as 0.234.)

5. A particle has spin $s = \frac{3}{2}$ and orbital angular momentum $l = 1$.
   (a) List all the allowed values of $\{m_s, m_l\}$. List also all the allowed values of $\{j, m_j\}$, where the vector operator $\widehat{\mathbf{J}}$ is the total angular momentum operator of the system. Check that the two lists are equal in length, and also that there are the same number of pairs with a given m_j in the second list as with a given $m_s + m_l$ in the first.
   (b) We can use either set of quantum numbers to define a basis. In the first case we denote the basis states by $|s\ m_s\rangle \otimes |l\ m_l\rangle$, and in the second case by $|l, s; j\ m_s\rangle$. By looking up the relevant Clebsch-Gordan coefficients, write the state $|1, \frac{3}{2}; \frac{1}{2}\ \frac{1}{2}\rangle$ as a sum of states from the first basis.

(c) The matrix representations of the angular momentum operators in the basis $\{|1\ 1\rangle, |1\ 0\rangle, |1\ {-1}\rangle\}$ are

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad L_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad L_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Use these to show that the state

$$\sqrt{\tfrac{2}{3}}\,|1\ 1\rangle \otimes |\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{1}{3}}\,|1\ 0\rangle \otimes |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

is an eigenfunction of $\widehat{\mathbf{J}}^2$ with eigenvalue $\frac{3}{4}\hbar^2$.

6. The states $\{|n\rangle\}$ are the eigenstates of a harmonic oscillator of mass m and oscillator frequency ω. Where appropriate use the length $x_0 = \sqrt{\hbar/(m\omega)}$ in your answers.

   (a) Use creation and annihilation operators to find the matrix elements $\langle m|\hat{x}|n\rangle$, $\langle m|\hat{p}|n\rangle$, $\langle m|\hat{x}^2|n\rangle$, $\langle m|\hat{p}^2|n\rangle$.

   (b) From the results of the previous part, find the uncertainty product $\Delta x \Delta p$ for a particle in the nth state, and comment on the result.

7. *This question is a little involved, but we will use the exact results obtained here to check our answers when we treat the same problem in perturbation theory later on.*
   Consider the new Hamiltonian corresponding to adding a linear potential to a harmonic oscillator, so that $\widehat{H}' = \dfrac{\hat{p}^2}{2m} + \dfrac{1}{2}m\omega^2\hat{x}^2 - \lambda\hat{x}$.

   (a) Rewrite the Hamiltonian to make clear that it is simply a new harmonic potential, and find expressions for the shift in the origin δ and the energy offset ΔE.

   (b) Show that the relationship between the creation and annihilation operators for the new system, $\hat{b}^\dagger$ and $\hat{b}$, and those for the original system $\hat{a}^\dagger$ and $\hat{a}$, is $\hat{b}^\dagger = \hat{a}^\dagger - \delta/(\sqrt{2}x_0)$ and $\hat{b} = \hat{a} - \delta/(\sqrt{2}x_0)$.. Use these to find the coefficients in the expansion of the new ground state $|0'\rangle$, annihilated by $\hat{b}$, in terms of the states of the original system $|n\rangle$, and check that the sum of their magnitudes squared is unity. (Hint: You will need to use the wave function to calculate the overlap of the old and new ground states $\langle 0|0'\rangle$, but all the other overlaps $\langle n|0\rangle$ can be reduced to this one with operator methods.)