PHYS20672 Complex Variables and Integral Transforms: Course Summary

The following is a short summary of the course, with an emphasis on the skills that you might be asked to demonstrate in an exam, and highlighting questions that require these skills.

- Differentiation of functions of complex variables; analytic (regular) functions, uniqueness of derivative and Cauchy Riemann equations; finding $u(x, y)$ given $v(x, y)$ and vice versa (also in plane polars, eg finding $u(r, \theta)$ given $v(r, \theta)$).
  qu. 10, 11, 14, 15

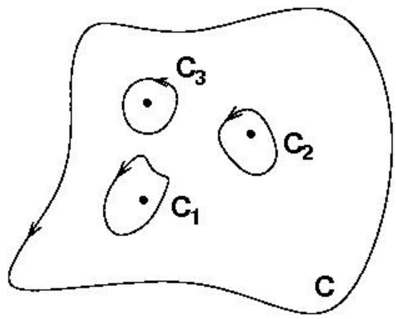
- Analytic functions as conformal mappings; u and v as solutions of Laplace's equation; finding solutions of Laplace's equation using a conformal mapping to map the boundaries to a simpler geometry (electrostatic and heat conduction problems).
  qu. 9, 13, 17-18

- Explicit integration along a path in the complex plane in Cartesian ($\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$) or polar ($\mathrm{d}z = ir\mathrm{e}^{i\theta}\mathrm{d}\theta$ for fixed r) coordinates. In particular for C a unit circle round the point $z = a$,

$$\oint_C \frac{1}{z-a}\mathrm{d}z = 2\pi i; \qquad \oint_C \frac{1}{(z-a)^n}\mathrm{d}z = 0 \quad \text{for integer } n > 1.$$

  Estimation Lemma. Path independence of integrals of analytic functions.
  qu 20, 21, 24

- Cauchy's theorem: $\oint_C f(z)\mathrm{d}z = 0$ if $f(z)$ is analytic on and inside C, hence we are free to deform the contour within any region in which $f(z)$ is analytic, or to replace contour C with a set of disjoint contours circling each of the regions of non-analyticity within C



  Most questions from qu. 22 onwards

- Cauchy's integral formulae:

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\mathrm{d}z \qquad \text{and} \qquad f^{(n)}(a) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}\mathrm{d}z$$

  In the light of the subsequent sections of the course, we can rephrase these to say that if $g(z) = f(z)/(z-a)^{n+1}$ with $f(z)$ non-zero and analytic around $z - a$, the residue of $g(z)$ at $z = a$ is $f^{(n)}(a)/n!$ - or just $f(a)$ if $n = 1$.
  qu. 23, 25 (and later ones)

- Taylor and Laurent series about arbitrary points and their radii of convergence as set by the position of poles. Laurent series about singularities and the classification of singularities (single and double poles etc). Note that the expressions for the Taylor and Laurent coefficients in terms of contour integrals are rarely used, but common MacClaurin expansions (of $\sin z$ etc) are.
qu. 28, 31, 32, 33, 34

- Residue theorem: for a meromorphic function,
$\oint_C f(z)\mathrm{d}z = 2\pi i$(Sum of residues at all poles inside C).
Methods of finding residues when the function isn't easily written as $f(z)/(z-a)^{n+1}$.
All of sheet 5.

- Calculation of real integrals such as $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$ by completing the contour in the complex plane via a line along which the integral can be done (typically a semicircle at infinity along which the integral vanishes); Jordan's Lemma for the case $f(x) = \mathrm{e}^{ikx}g(x)$; treatment of poles on the real axis and of branch cuts; summation of series.
qu. 35, 36, 38-42

- Fourier Transforms and inverse, calculation via contour integration including appropriate use of Jordan's Lemma; convolution theorem.
qu. 44-48

- Laplace Transforms; inverse Laplace transforms via lookup tables; use of shift theorems and treatment of derivatives etc in both.
qu. 51-53

- Use of Laplace Transforms to solve differential equations.
qu. 54-57

- Use of Bromwich integral to calculate inverse Laplace transforms: for meromorphic $F(s)$, $f(t)$ given by sum of residues of $\mathrm{e}^{st}F(s)$.
qu. 58 (and notes at the end of Examples 6).