**PHYS20672 Complex Variables and Integral Transforms: How to find residues**

Cauchy's residue theorem states that for a meromorphic function $f(z)$ and a curve $C$ which encloses a number of poles of $f(z)$, the residue theorem states that

$$\oint_C f(z)\mathrm{d}z = 2\pi i(\text{Sum of residues at all poles inside } C)$$

So calculating countour integrals in such a case reduces to calculating the residues of $f(z)$ at the relevant poles.

First, we have to find the poles. Usually this is obvious; any zero of the denominator is a potential pole, though we need to check the whether the numerator also has a zero at that point before we can be sure. We then use one of the following rules to find the residue at a pole at $z = a$.

1. If the function has the form $f(z) = \dfrac{g(z)}{z-a}$ where $g(z)$ is analytic in the immediate vicinity of $z = a$, Cauchy's integral formula tells us that the residue is $g(a)$. (Note the change of notation: in the usual statement of the formula, $f(z)$ refers to the analytic part, not the whole integrand—what we are calling $g(z)$ here. Note also that $g(z)$ can be quite complicated, and can have terms in the denominator so long as they don't vanish at $z = a$).

2. Similarly if the function has the form $f(z) = \dfrac{g(z)}{(z-a)^{n+1}}$ where $g(z)$ is analytic, the residue is $g^{(n)}(a)/n!$. This is the $n$th term in the Taylor series of $g(z)$ about $z = a$, so if we know that series, we use rule 4 instead.

3. If $f(z) = \dfrac{g(z)}{h(z)}$ where $h(z)$ and $g(z)$ are analytic in the immediate vicinity of $z = a$, and $g(a) \neq 0$ but $h(z)$ has a simple zero at $z = a$, then we use $b_1 = \dfrac{g(a)}{h'(a)}$. (The proof follows from rule 5, using L'Hôpital's rule.)

4. For $f(z)$ consisting of a number of analytic parts in the numerator and denominator, we can use Taylor series for each of them separately and then combine, using if necessary a geometric or binomial series to expand the denominator. If quadratic or higher terms are needed, note

$$\begin{aligned}\Big(1+\sum_{n=1} a_n z^n\Big)^{-n} &= 1 - n\sum_{n=1} a_n z^n + \tfrac{1}{2}n(n+1)\Big(\textstyle\sum_{n=1} a_n z^n\Big)^2 + \ldots \\ &= 1 - na_1 z + \big(\tfrac{1}{2}n(n+1)a_1^2 - na_2\big)z^2 + \ldots\end{aligned}$$

   If trig functions are involved and the residue is at a point $a$ other than at $z = 0$, first shift the argument to $w = z - a$.

5. *The following expressions cannot simply be evaluated by setting $z = a$, but instead require the limit $z \to a$ to be taken. For that reason their practical use usually reduces to rules 3&4 above.*
   If we know that $f(z)$ has a simple pole at $z = a$, we can use $b_1 = \lim_{z\to a}(z-a)f(z)$. (This

basically comes from rule 1, defining $g(z) = (z-a)f(z)$ and knowing that it must be analytic since the pole of $f(z)$ is simple.) Similarly if we know that $f(z)$ has a pole of order $n$ at $z=a$, $b_1 = \dfrac{1}{(n-1)!}\lim_{z\to a}\dfrac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\Big((z-a)^n f(z)\Big)$. Note that we will not go wrong if we overestimate the order of the pole in this formula (ie choose n greater than the actual order of the pole), but we will give ourselves unnecessary work.

Examples:

- The residue of $\dfrac{\sin z}{z-2}$ at $z=2$ is $\sin 2$ (rule 1).
- The residue of $\dfrac{\sin z}{z^2-4}$ at $z=2$ is $\left.\dfrac{\sin z}{z+2}\right|_{z=2} = \dfrac{\sin 2}{4}$ (rule 1).
- The residue of $\dfrac{z^3}{(z-2)^2}$ at $z=2$ is $3z^2|_{z=2} = 12$ (rule 2).
- The residue of $\dfrac{\sin z}{z^6}$ at $z=0$ is $1/5!$ (rule 4).
- The residue of $\dfrac{1}{z^3+i}$ at $z=i$ is $\left.\dfrac{1}{3z^2}\right|_{z=i} = -\dfrac{1}{3}$ (rule 3). This is much faster than factorizing $z^3+i = (z+i)(z-\mathrm{e}^{-5\pi i/6})(z-\mathrm{e}^{-\pi i/6})$ then using rule 1.
- The residue of $\dfrac{1}{\cosh z - 2\sinh z - 1}$ at $z=0$ is $\left.\dfrac{1}{\sinh z - 2\cosh z}\right|_{z=0} = -\frac{1}{2}$ (rule 3; since the derivative of the denominator does not vanish at $z=0$ the pole is simple.)
- The residue of $\dfrac{1}{z^3\sin^2 z}$ at $z=0$ is found from (rule 4)

$$\begin{aligned}(\sin z)^{-2} &= z^{-2}\Big(1-\frac{z^2}{6}+\frac{z^4}{5!}-\dots\Big)^{-2} = z^{-2}\Big(1-2\Big(-\frac{z^2}{6}+\frac{z^4}{5!}\Big)+3\Big(-\frac{z^2}{6}\Big)^2-\dots\Big)\\ &= z^{-2}\Big(1+\frac{z^2}{3}+\frac{z^4}{15}-\dots\Big)\end{aligned}$$

  so that the residue is $1/15$.
- The residue of $z^2\mathrm{e}^{1/z}$ at $z=0$ is $1/6$ (the coefficient of $(1/z)^3$ in the expansion of $\mathrm{e}^{1/z}$) (rule 4).