PHYS20672 Complex Variables and Integral Transforms: Examples 5

35. Evaluate the following integrals using contour integration:

$$a)\ \int_{-\infty}^{\infty} \frac{1}{1+x^4}\mathrm{d}x \qquad b)\ \int_{-\infty}^{\infty} \frac{x^4}{1+x^8}\mathrm{d}x \qquad c)\ \int_{-\infty}^{\infty} \frac{1}{(x^2-2x+5)^2}\mathrm{d}x$$

36. Evaluate the following integrals using contour integration; in each case check that the conditions for Jordan's lemma to hold are satisfied:

$$a)\ \int_{-\infty}^{\infty} \frac{x\sin x}{(1+x^2)^2}\mathrm{d}x \qquad b)\ \int_{-\infty}^{\infty} \frac{\sin \pi x}{1+x+x^2}\mathrm{d}x$$

What would we get in each case if we replaced $\sin x$ with $\cos x$?

37. Let a be a real number, and C be the (open) contour round a semicircle of radius ϵ, centred on the point $z = a$, starting and ending on the real axis and taken anticlockwise. Consider the integral round C of $(z-a)^n$ where n is an integer which can be positive, zero or negative. Show that the integral vanishes for odd n, except for $n = -1$, and is πi for $n = -1$. Show also that for even n, the limit as $\epsilon \to 0$ is zero if $n > -1$ and undefined if $n < -1$. Hence show that if $f(z)$ has a simple pole at $z = a$, the integral on C is

$$\lim_{\epsilon\to 0}\int_C f(z)\mathrm{d}z = \frac{1}{2}\oint f(z)\mathrm{d}z = i\pi b_1^{z=a} \qquad \text{where } b_1^{z=a} = \lim_{z\to a}(z-a)f(z).$$

Evaluate the following, where in each case C is the small semicircle round the pole described above:

$$a)\ \lim_{\epsilon\to 0}\int_C \frac{\mathrm{e}^z}{z}\mathrm{d}z \qquad b)\ \lim_{\epsilon\to 0}\int_C \frac{z^2-2z+1}{z+1}\mathrm{d}z \qquad c)\ \lim_{\epsilon\to 0}\int_C \frac{1-\mathrm{e}^z}{z^2}\mathrm{d}z$$

38. The following integrals all have poles on the real axis. Find the Cauchy principal value using contour integration; where appropriate check that the conditions for Jordan's lemma to hold are satisfied.

$$a)\ \int_{-\infty}^{\infty} \frac{1}{(x-2)(x^2+1)}\,\mathrm{d}x \qquad b)\ \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2-4)}\,\mathrm{d}x \qquad c)\ \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x$$

Hint: for (c), use $\sin^2 x = \frac{1}{2}\mathrm{Re}\,(1-e^{2ix})$

39. Evaluate

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ixt}}{\alpha + ix}\,\mathrm{d}x$$

where $\alpha > 0$ but t can be positive or negative. (Hint – consider positive and negative t separately.)

40. Choose a suitable contour to evaluate

$$\int_0^{\infty} \frac{\sqrt{x}}{(x+1)^2}\,\mathrm{d}x$$

41. Use an appropriate contour integral of the functions suggested to prove the following series:

a)

$$f(z) = \frac{\cot z}{z^4}; \qquad \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots = \frac{\pi^4}{90}$$

b)

$$f(z) = \frac{1}{z^5 \cos z}; \qquad \frac{1}{1^5} - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \ldots = \frac{5\pi^5}{1536}$$

42. By considering a change of variable $w = 1/z$, and defining $g(w) = f(1/w)$, show that

$$\oint_C f(z)\mathrm{d}z = \oint_{C'} \frac{g(w)}{w^2}\mathrm{d}w$$

where C' is the curve on the w plane corresponding to the curve C in the z plane, but traversed in the conventional (anticlockwise) direction. For instance if C is the circle $|z| = R$, C' is the circle $|w| = 1/R$. (Pay attention to the sign!)

Hence show that the sum of the residues of $f(z)$ within C must equal the sum of the residues of $g(w)/w^2$ within C'. Verify this explicitly for $f(z) = 1/(z^2 - 3z + 2)$ and C being the circle $|z| = R$ for $R = 1/2$, $3/2$ and $5/2$.