PHYS20672 Complex Variables and Integral Transforms: Examples 2

10. Using the definition of the derivative, differentiate the following (or show that the derivative doesn't exist):

$$a)\ z^3 + z^2 \qquad b)\ 1/z \quad (\text{for } z \neq 0) \qquad c)\ |z|^2$$

For each case, identify $u(x, y)$ and $v(x, y)$ and determine where, if anywhere, the Cauchy-Riemann equations are satisfied.

11. Assuming the usual rules for differentiation of real functions (eg $\frac{\mathrm{d}(\sin x)}{\mathrm{d}x} = \cos x$) show that

$$a)\ \frac{\mathrm{d}(\sin z)}{\mathrm{d}z} = \cos z \qquad b)\ \frac{\mathrm{d}(\ln z)}{\mathrm{d}z} = \frac{1}{z}$$

In each case find the region in which the Cauchy-Riemann equations are satisfied.

12. Let $f(z) = u+iv$ and $g(z) = s+it$ be analytic functions of z. Consider the function $g(w)$ where $w = u + iv$ and write the Cauchy-Riemann equations in terms of these variables (*ie* using $\partial s/\partial u$ etc). Hence show that the function $g\big(f(z)\big)$ is an analytic function of z, by showing that the Cauchy-Riemann eqations for $\partial s/\partial x$ etc are satisfied.

13. i) Prove that if $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations, they are also "harmonic", that is they satisfy Laplace's equation.

ii) Prove that the form of the Cauchy-Riemann equations in polar coordinates below follows from the Cartesian form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta},$$

iii) Prove that if $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations, $f = u + iv$ is a function only of z and not of $\overline{z}$. (Hint: write $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/(2i)$, and show that $\partial f/\partial \overline{z}|_z = 0$ and $\partial f/\partial z|_{\overline{z}} = \mathrm{d}f/\mathrm{d}z$.)

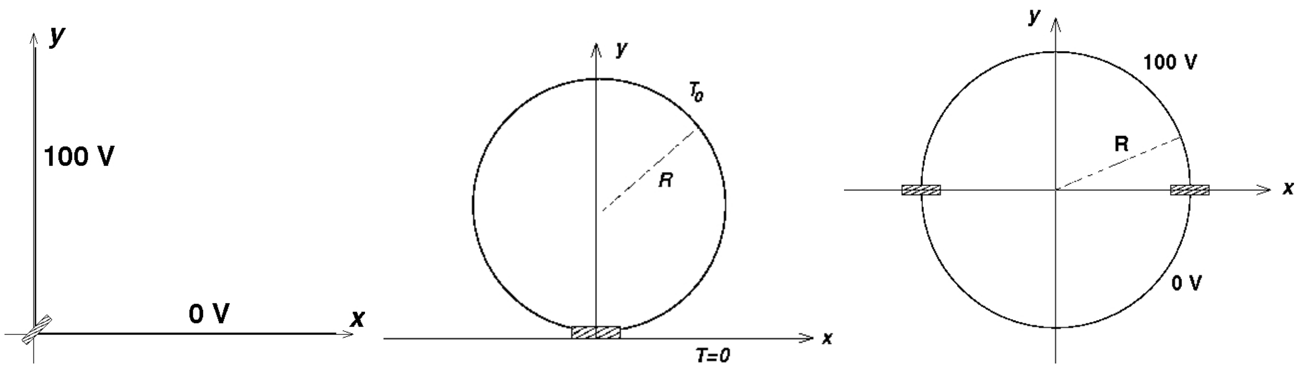
14. $u(x, y) = 2xy$ is the real part of an analytic function $f(z)$. Using the Cauchy-Riemann equations, find the conjugate function $v(x, y)$ which is the imaginary part. Hence construct $f(z)$.

15. An analytic function $f(z)$ has imaginary part $v(x, y) = y\mathrm{e}^x \cos y + x\mathrm{e}^x \sin y$. Show that $v(x, y)$ is harmonic, and find the corresponding real part of $f(z)$. Express $f(z)$ in terms of z.

16. In general, when we change from one set of (real) coordinates x, y to another set u, v, the new area element $\mathrm{d}u\,\mathrm{d}v$ is equal to $\mathcal{J}\,\mathrm{d}x\,\mathrm{d}y$ where the Jacobian $\mathcal{J}$ is defined as

$$\mathcal{J} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix},$$

where $|\ldots|$ represents the determinant. Show that if $f(z) = u + iv$ is analytic, $\mathcal{J} = |\mathrm{d}f/\mathrm{d}z|^2$ (where here $|\ldots|$ represents the magnitude!)



The next three questions refer to these three figures, which also show the coordinates to use in each case.

17. Two semi-infinite metal sheets are at right angles to each other; one (in the xz plane) is held at 0 Volts and the other (in the yz plane) is held at 100 Volts. We want to find the potential in the region $x > 0$, $y > 0$. Argue that the potential is a function of x and y only. (Henceforth z will refer to $x + iy$ as usual.)

    Use the method of conformal mapping, with the transformation $Z = \ln z$, to show that the two plates map into a parallel plate capacitor with plates at $Y = 0$ and $Y = \pi/2$. Find the potential in terms of $\{X, Y\}$ and hence in terms of $\{x, y\}$, verifying that it obeys the boundary conditions in the original geometry. Sketch some equipotentials and field lines.

18. A hot cylinder of radius R ($T = T_0$) rests on, but is insulated from, an infinite cold plate ($T = 0^o$ C). Use the transformation $Z = R^2/z$ to map the surface of the cylinder and the plate to two infinite parallel plates at $Y = -R/2$ and $Y = 0$ respectively (take the point of contact to be $z = 0$). Hence show that the temperature distribution in the region above the plane and outside the cylinder in the original problem is $T = 2T_0Ry/(x^2 + y^2)$ and sketch some isotherms and lines of heat flow.

19. Consider the conformal mapping $Z = \ln\left(\dfrac{R+z}{R-z}\right)$.

    For points with $|z| < R$, show that

    $$Y = \arctan\left(\frac{y}{R+x}\right) + \arctan\left(\frac{y}{R-x}\right) = \arctan\left(\frac{2yR}{R^2 - x^2 - y^2}\right)$$

    Find the shape of lines of constant Y in the xy plane, for $|Y| < \pi/2$.

    By taking the limit as $|z| \to R$, show that the semi-circles with $|z| = R$ above and below the x-axis correspond to $Y = \pi/2$ and $Y = -\pi/2$ respectively.

    A capacitor consists of two half-cylinders, radius R, which are insulated from one another where they nearly touch. The upper half is held at 100 V and the lower half at 0 V. Use the results above to show that the potential between the half-cylinders is

    $$\phi(x, y) = \left[50 + \frac{100}{\pi}\arctan\left(\frac{2yR}{R^2 - x^2 - y^2}\right)\right] \text{ V}.$$

    Sketch the equipotentials, and without further calculation add an educated guess at the field lines to your sketch.