PHYS30201 Advanced Quantum Mechanics:
Particle Data Group Clebsch-Gordan coefficients

In a system with two contributions to angular momentum j_1 and j_2, Clebsch-Gordan coefficients are used to write states good of total angular momemtum J and z-component M, $|j_1, j_2; J\ M\rangle$, in terms of the basis $\{m_1, m_2\}$, $|j_1\ m_1\rangle \otimes |j_2\ m_2\rangle$:

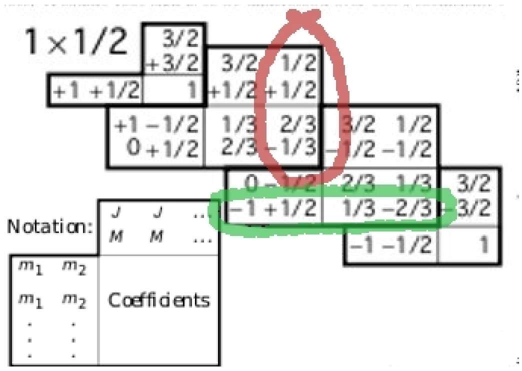
$$|j_1, j_2; J\ M\rangle = \sum_{m_1 m_2} \langle j_1\ m_1; j_2\ m_2|J\ M\rangle \Big(|j_1\ m_1\rangle \otimes |j_2\ m_2\rangle\Big) \qquad \text{and}$$

$$|j_1\ m_1\rangle \otimes |j_2\ m_2\rangle = \sum_{JM} \langle j_1\ m_1; j_2\ m_2|J\ M\rangle\, |j_1, j_2; J\ M\rangle$$

where the numbers denoted by $\langle j_1\ m_1; j_2\ m_2|J\ M\rangle$ are the Clebsch-Gordan coefficients; they vanish unless $j_1 + j_2 \geq J \geq |j_1 - j_2|$, and $m_1 + m_2 = M$. There is a conventional tabulation which can be found in various places including the Particle Data Group site, but the notation takes some explanation.

There is one table for each j_1, j_2 pair. Along the top are possible values of ${}_{JM}$ and at the left are possible values of $m_1\ m_2$. For compactness the numbers in the blocks are the coefficients squared times their sign; thus $-\frac{1}{2}$ stands for $-\sqrt{\frac{1}{2}}$.

As an example consider the table for coupling $j_1 = 1$ and $j_2 = \frac{1}{2}$ to get $J = \frac{3}{2}$ or $\frac{1}{2}$. In red the coefficients of $|1\ 1\rangle \otimes |\frac{1}{2}\ {-\frac{1}{2}}\rangle$ and $|1\ 0\rangle \otimes |\frac{1}{2}\ \frac{1}{2}\rangle$ in $|1\ \frac{1}{2}; \frac{1}{2}\ \frac{1}{2}\rangle$ are highlighted

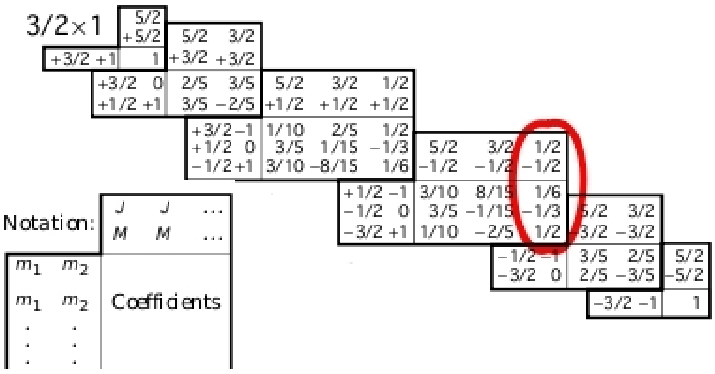


$$|1, \tfrac{1}{2}; \tfrac{1}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|1\ 1\rangle \otimes |\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{1}{3}}|1\ 0\rangle \otimes |\tfrac{1}{2}\ \tfrac{1}{2}\rangle.$$

in green are the components for the decomposition

$$|1\ \ {-1}\rangle \otimes |\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|1, \tfrac{1}{2}; \tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{2}{3}}|1\ \tfrac{1}{2}; \tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle.$$

Or for coupling $j_1 = \frac{3}{2}$ and $j_2 = 1$:



giving for example

$$|1, \tfrac{3}{2}; \tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle = \sqrt{\tfrac{1}{6}}\,|\tfrac{3}{2}\ \tfrac{1}{2}\rangle \otimes |1\ {-1}\rangle - \sqrt{\tfrac{1}{3}}\,|\tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle \otimes |1\ 0\rangle + \sqrt{\tfrac{1}{2}}\,|\tfrac{3}{2}\ {-\tfrac{3}{2}}\rangle \otimes |1\ 1\rangle$$

If instead one wants $j_1 = 1$ and $j_2 = \frac{3}{2}$, we use the relation

$$\langle j_2\, m_2; j_1\, m_1 | J\, M\rangle = (-1)^{J-j_1-j_2} \langle j_1\, m_1; j_2\, m_2 | J\, M\rangle$$

Note that table of Clebsch-Gordan coefficients are given for states of j_1 and j_2 coupling up to total J. But as j is a generic angular momentum, that covers s and l coupling to j, or s_1 and s_2 coupling to S etc.