The Harmonic Oscillator Without Tears

Summary: Operator methods lead to a new way of viewing the harmonic oscillator in which quanta of energy are primary.

We are concerned with a particle of mass m in a harmonic oscillator potential $\frac{1}{2}kx^2 \equiv \frac{1}{2}m\omega^2x^2$ where ω is the classical frequency of oscillation. The Hamiltonian is

$$\widehat{H} = \frac{\hat{p}^2}{2m} + \tfrac{1}{2}m\omega^2\hat{x}^2$$

and we are going to forget that we know what the energy levels and wavefunctions are.

If we define

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0} + i\frac{x_0}{\hbar}\hat{p}\right) \qquad \text{and} \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0} - i\frac{x_0}{\hbar}\hat{p}\right)$$

where $x_0 = \sqrt{\hbar/m\omega}$ we can prove the following:

- $\hat{x} = (x_0/\sqrt{2})(\hat{a}^\dagger + \hat{a})$; $\hat{p} = (i\hbar/\sqrt{2}x_0)(\hat{a}^\dagger - \hat{a})$
- $[\hat{x}, \hat{p}] = i\hbar \Rightarrow [\hat{a}, \hat{a}^\dagger] = 1$
- $\widehat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$
- $[\hat{a}, H] = \hbar\omega\,\hat{a}$ and $[\hat{a}^\dagger, H] = -\hbar\omega\,\hat{a}^\dagger$
- Assume we know one eigenstate of $\widehat{H}$, $|n\rangle$, with energy E_n (notation to be explained later). Since $\langle n|\hat{a}^\dagger\hat{a}|n\rangle = \langle \hat{a}n|\hat{a}n\rangle \geq 0$, $E_n \geq \frac{1}{2}\hbar\omega$.
- Using the commutators above, we find that $\hat{a}|n\rangle$ is another eigenstate with energy $E_n - \hbar\omega$ and $\hat{a}^\dagger|n\rangle$ is another eigenstate with energy $E_n + \hbar\omega$.
- We denote the state of lowest energy $|0\rangle$ (*not* the null state!). Since there is no lower state this must be an exception to the rule that $\hat{a}$ takes us to a state with lower energy, so in this case the equation $[\hat{a}, H]|0\rangle = \hbar\omega\hat{a}|0\rangle$ must be satisfied by $\hat{a}|0\rangle = 0$ (where 0 *is* the null state or vacuum).
- The energy of state $|n\rangle$ is therefore $E_0 + n\hbar\omega$ and the notation becomes clear: $|n\rangle$ is the nth excited state.
- The commutation relation $[\hat{a}, \hat{a}^\dagger]|n\rangle = |n\rangle$ requires the following normalisations:

  $$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \qquad \text{and} \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

- $\hat{a}^\dagger\hat{a}$ is a "number operator", since $\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$. Thus we have

  $$\widehat{H}|n\rangle = (n + \tfrac{1}{2})\hbar\omega|n\rangle$$

  and the ground state energy E_0 is $\frac{1}{2}\hbar\omega$.

- Writing $\phi_0(x) \equiv \langle x|0\rangle$, from $\langle x|\hat{a}|0\rangle = 0$ we obtain $\mathrm{d}\phi_0/\mathrm{d}x = -(x/x_0^2)\phi_0$ and hence

$$\phi_0 = (\pi x_0^2)^{-1/4} \mathrm{e}^{-x^2/2x_0^2}$$

  (where the normalisation has to be verified separately). This is a much easier differential equation to solve than the one which comes direct from the Schrödinger equation!

- The wave function for the n-th state is

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{x}{x_0} - x_0 \frac{\mathrm{d}}{\mathrm{d}x} \right)^n \phi_0(x) = \frac{1}{\sqrt{2^n n!}} H_n(\tfrac{x}{x_0}) \phi_0(x)$$

- The Hermite polynomials are $H_0(z) = 1$; $H_1(z) = 2z$; $H_2(z) = 4z^2 - 2$; $H_3(z) = 8z^3 - 12z$; $H_4(z) = 16z^4 - 48x^2 + 12$

This formalism has remarkably little reference to the actual system in question – all the parameters are buried in x_0. What is highlighted instead is the number of quanta of energy in the system, with $\hat{a}$ and $\hat{a}^\dagger$ annihilating or creating quanta. Exactly the same formalism can be used in a quantum theory of photons, where the oscillator in question is just a mode of the EM field.

- **Shankar ch 7.4-5**
- **Mandl ch 12.5**
- **Gasiorowicz ch 6.2-3**

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